Biquadric Fields:
Equipping Finite Projective Spaces With “Metric” Structure

Master’s Thesis

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Contents

1 A New Guise for Spacetime

"Suppose within the girdle of these walls
Are now confin'd two mighty monarchies,
Whose high upreared and abutting fronts
The perilous narrow ocean parts asunder[.]"

(William Shakespeare, The Life of Henry the Fifth, Prologue)

Gravity seems to be the dominating interaction that shapes the universe on very large length scales, while on very small length scales *quantum matter* is prevailing. The former is pretty well described by the *General Theory of Relativity (GR)* while the latter is by *Quantum Field Theory (QFT)*. Both theories have their own conceptual problems [Pad98, Pen05]. But more importantly both are articulating themselves in two very distinct languages using contradictory concepts. The role of *time* in GR is a dynamical object of the theory itself and the *very same time* serves only as a parameter in QFT[1]. Another point of friction between both theories is that GR is inherently local, while QFT is inherently global. This is maybe the most fundamental question of modern physics: how to combine a framework where the geometry itself is a dynamical object with the language of QFT or how to replace both by a profoundly new theory. Naive attempts to quantize the energy-momentum tensor in order to quantize the Einstein field equations fail directly due to nonrenormalizability that already occurs for two-loop terms [Pad98, Gor85]. These divergencies could only be taken care of by applying "unnatural" ad hoc cut-offs.

The most prominent theories trying to deal with these issues are string theory and loop quantum gravity. But again both have several conceptual problems and there is no breakthrough result or prediction that has been proven yet. Hence there is an ongoing search for new theories. Some of them try a *bottom-up* approach of starting with a discrete (and sometimes additionally with a finite) structure as the underlying geometry instead of the *top-down* approach of quantizing a continous theory of space-time. Prominent representatives for bottom-up approaches are the twistor-theory (see p. 958 et seq. in [Pen99]), causal sets [Bom87], causal dynamical triangulations [Ar98] or to employ *finite fields* [Ahm65]. Peskin and Schroeder also speculate (at the end of their well-known standard textbook on QFT, p. 798 in [Pes95]):

> "At the scale where quantum fluctuations of the gravitational field are important, we must expect *profound* changes in physics. If these changes occur within the context of quantum field theory, they will at the least

[1] Even in QFT on a curved spacetime there is no complete recoupling of the quntum matter curving spacetime and thus the role of time is still different.

> entail fluctuating spacetime geometry and topology. But it seems *equally probable* that quantum field theory will actually break down at this scale, with continous spacetime replaced by a new *discrete or nonlocal geometry*."

Within this thesis I neither undertook the mammoth task of developing a new theory nor did I solve a problem of a yet established theory, but I exploratively analyzed a new notion that seems promising for utilizing finite *projective geometries* in a bottom-up approach to a discrete spacetime geometry.

For finite fields there is no global order relation and hence volumens of certain dimensions seem to be absurd; though there are several mechanisms to define at least partially ordered *subsets*. While constructing geometries over these finite fields — even after solving the order problem another problem — still remains: in all geometries used in common physics, the location of points that share the same *distance* to a given point is encoded in a mathematical object called *quadric* (which is a cone section), like one does with the metric in GR. This object can be used to define "unit lengths" in *in each direction* within all tangent spaces of the spacetime. But quadrics over finite fields do not serve with enough points arranged in such a way that there is a quadric point in every direction.

The main notion that has been explored in this master's thesis is a pair of quadrics that serves to fulfill this task of serving with a point on the quadric in each direction: *biquadrics*. They are looked at within finite projective geometries, because we want to restrict the assumptions on the spacetime geometry to the bare minimum in a way that is also as symmetric as possible concerning different geometrical objects while assuming a finite and discrete world. It has to be shown first that this "educated and aesthetic guess" is of relevance; but in case it is, the notion of biquadric fields seems very likely to me to become valueable.

The results so far [Ale12, Win12] have been answering how to describe these biquadrics for a specially coordinatized, so-called, center point and its polar[2] and biquadrics centered "around" other points have been solely produced by transforming these biquadrics for the special center point and polar to other center points and polars (case-by-case) in order to produce *biquadric fields* that have all points of the finite projective geometry as center points. Biquadric fields are thus analogous to the metric tensor field. Nevertheless no point is distinguished within projective geometries and thus I tried to find a more general form of matrix pairs parameterizing biquadrics with respect to all center points and polars in order to understand them better. The results so far had also only been shown for the two-dimensional projective plane and I have proven the general form for arbitary dimensions. In order to do so I had to find an explicite form[3] of so-called *dehomogenizations*, *homogenizations*, and *affinities* with respect to an arbitrary hyperplane at infinity, i.e., how to coordinatize affine points

[2] The special center point is $p_c = (0, 0, 1)^t$ and its respective polar is $\mathrm{pol}_{p_c} = (0, 0, 1)^t$.

[3] "Explicite form" refers to a parameterized affine-linear matrix expression.

after "slicing" the projective geometry into an affine part and a hyperplane at infinity and how the automorphisms of this affine part are parameterized explicitely.

Furthermore, so far only quadric fields and not *bi*quadric fields had been simulated. In order to do so I wrote a C++ library, called `libgalois` and used it to produce random biquadric fields for various prime numbers and searched for two types of flat biquadric fields in order to test how the notion of biquadric fields can serve to encode *curvature* in finite projective geometries. I also produced all biquadrics for a small finite projective geometry, namely that coordinatized over the Galois field $\mathbb{F}_3$. Within the same geometry I found two flat states; one that is flat within an affine plane but not on the respective line at infinity while the other state is flat within the whole projective geometry. The analysis lead to a refinement of the notion of a biquadric. Even if the application to a spacetime description becomes indefensible, the generalized (parameterized) forms of several objects might save a lot of simulation time if the library is possibly used to simulate something completely different.

The whole project lead to some interesting questions and so far the notion of biquadric fields has not been ruled out in order to equip finite projective geometries with structure. Quite the contrary might be true: it could serve as an essential object within a new approach to utilize finite projective geometries as the underlying spacetime of a mathematical description of the world; but future research has to show that.

2 Biquadric Fields on Finite Projective Spaces

> "Categories such as time, space, cause, and number represent the most general relations which exist between things; surpassing all our other ideas in extension, they dominate all the details of our intellectual life. If humankind did not agree upon these essential ideas at every moment, if they did not have the same conception of time, space, cause, and number, all contact between their minds would be impossible..."
>
> (Émile Durkheim, *Les formes élémentaires de la vie religieuse)*

The mathematical structures supposed to serve as the underlying spacetime in this investigation are projective spaces over finite fields. In the following chapter the mathematics needed to deal with the notions of finite projective spaces over so-called Galois fields will be developed in the general d-dimensional case for them being available as tools for the two-dimensional case of the projective planes. The former because the theoretical results of this master thesis are formulated for the general case and the latter because the simulations have been executed for projective planes so far. Finally these theoretical results on biquadric fields, the central objects of this work, are developed.

2.1 Projective Spaces over Finite Fields

There are two major ways to construct a geometric space: a synthetic geometrical approach (by axioms) or an analytical one (involving coordinates). Both perspectives yield different insights; but the latter serves with the important calculation techniques. Therefore both ways will be presented.

2.1.1 The Synthetic Way to Geometries: Projective Spaces by Axioms

The Basic Definition

Geometry in all its varieties models the *relations* of different objects. Points, lines, and surfaces are just examples of such objects while nodes and vertices of networks are others. The natural relation they have is whether they lie in each other in some way or not. But their mutual relations do not have to be in this natural way. To render this basic property of geometries more precisely and at the same time abstract it, the following definition — that builds the core of synthetic geometry — seems to be appropriate:

Definition 2.1. *Geometry $\mathscr{G}$ (see p.* 1 *in [Beu98])*

A geometry $\mathscr{G} = (\Omega, \mathscr{I})$ is an ordered pair of a set Ω and a binary relation $\mathscr{I} \subset \Omega \times \Omega$) on Ω — called incidence relation[1] *— that is both*

- *symmetric:* $\forall \omega_1, \omega_2 \in \Omega : (\omega_1, \omega_2) \in \mathscr{I} \Leftrightarrow (\omega_2, \omega_1) \in \mathscr{I}$ *and*
- *reflexive:* $\forall \omega \in \Omega : (\omega, \omega) \in \mathscr{I}$.

It is meaningful to base a geometry on a *set*, because manipulating grouped objects in a well defined way is exactly applying the notions of set theory. So Ω is simply the set of all objects that are relevant for the geometrical considerations one wants to encounter with the geometry $\mathscr{G}$.

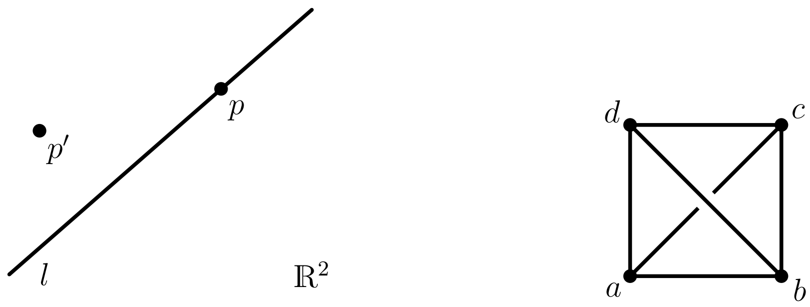
Including a *relation* into the structure makes sense in so far that it establishes a basis for talking about the relations of the geometrical objects in a literal sense. The name "*incidence relation*" is meaningful because in most cases this relation has to do with objects — or at least parts of them — *lying in* each other in a set theoretical sense. Stipulating this relation to be *reflexive* enables one to include the special case of an object lying in itself and the *symmetry* is meaningful because if object A is somehow in object B at least some part of object B has to lie in object A as well (roughly speaking). They have something in common; they are *related.* So at least in some sense they are *incident* and often they are literally.

Example 2.2. *Geometries*

- *Maybe the most accessible example is the two-dimensional Euclidean plane $\boldsymbol{E}^2$ where the set Ω is simply the set of all points and lines in the $\mathbb{R}^2$ with the incidence relation $\mathscr{I}$, such that a point p is incident with a line ℓ if and only if the point lies on the line ($p \in \ell$) and such that each object is incident with itself (see Fig. 2.1(a)).*
- *The set $\Omega = \{a, b, c, d, \{a,b\}, \{b,c\}, \{c,d\}, \{d,a\}, \{a,c\}, \{b,d\}\}$ is resembling the "sceleton" of a square with its four corners a, b, c, d, all edges $\{a,b\}$, $\{b,c\}$, $\{c,d\}$, $\{d,a\}$, and diagonals $\{a,c\}$, $\{b,d\}$ together with the containedness incidence relation*[2] *is a geometry $\mathscr{G}$ (see Figure 2.1(b)). The immersion into the plane of the page it is printed on might be misleading, because really the discrete*

[1] $\omega_1 \mathscr{I} \omega_2$ expresses that ω_1 is incident with ω_2, i.e., the tupel (ω_1, ω_2) is an element of $\mathscr{I}$.

[2] The incidence relation reads explicitly: $\mathscr{I} = \{(a,a), (b,b), (c,c), (d,d), (\{a,b\},\{a,b\}), (\{b,c\},\{b,c\}), (\{c,d\},\{c,d\}), (\{d,a\},\{d,a\}), (a,\{a,b\}), (\{a,b\},a), (b,\{a,b\}), (\{a,b\},b), (b,\{b,c\}), (\{b,c\},b), (c,\{b,c\}), (\{b,c\},c), (c,\{c,d\}), (\{c,d\},c), (d,\{c,d\}), (\{c,d\},d), (d,\{d,a\}), (\{d,a\},d), (a,\{d,a\}), (\{d,a\},a), (a,\{a,c\}), (\{a,c\},a), (c,\{a,c\}), (\{a,c\},c), (b,\{b,d\}), (\{b,d\},b), (d,\{b,d\}), (\{b,d\},d)\}$. This longish explicit form makes one feel for the need of a closed form for the incidence relations expressing set theoretical containedness. This explicit form will be given below in equation 2.6.



(a) The Euclidean plane $\mathbf{E}^2$ is a geometry $\mathscr{G}$. The elements of Ω are all points and lines of the $\mathbb{R}^2$. The incidence relation $\mathscr{I}$ is given by set theoretical containedness. The points and line in the figure are just examples ($p\mathscr{I}l$).

(b) A discrete square as another example for a geometry $\mathscr{G}$. All the elements of Ω are shown in this figure (different then in Figure (a)). The lines just symbolize that, e.g, $\{a, c\}$ is an object in the geometry; it is no continous line! The incidence relation is again encoding containedness.

Figure 2.1: The Euclidean plane (a) and a discrete square (b) are examples of geometries $\mathscr{G}$ as defined in Definition 2.1 according to Example 2.2.

square is meant. There are no other points on the lines than the two given above respectively (therefore the diagonal is displayed disconnected in Figure 2.1(b), because there is no intersection).

Characterizing Geometries

This notion of a geometry $\mathscr{G}$ describes an incredibly large category of structures. In order to apply this notion fruitfully one wants to have some characteristic properties of those synthetic geometries in order to classify them. A notion that enables one to do so is the notion of a "*flag*" and especially the one of a "*maximal flag*":

Definition 2.3. *Flag $\mathscr{F}$ (see p. 3 in [Beu98])*

A flag $\mathscr{F} \subset \Omega$ is a set of mutually incident objects:

- $\mathscr{F}$ *flag* $:\Leftrightarrow \forall f, g \in \mathscr{F} : f \mathscr{I} g$.

It is maximal if there is no non-empty set $\omega \in \Omega$ that if unified with the flag $\mathscr{F}$ yields another flag $\mathscr{F}'$; formally:

- $\mathscr{F}$ *maximal flag* $:\Leftrightarrow \neg\exists \omega \in \Omega \backslash \mathscr{F}, \omega \neq \emptyset : \mathscr{F}' = \mathscr{F} \cup \{\omega\}$ *flag.*

This notion of a maximal flag enables one to select geometries that have maximal flags that all have the same number of elements and group the elements of these typical maximal flags in different *types*.

Definition 2.4. *Rank r (see p. 4 in [Beu98])*

If there is a partition of Ω into mutually disjunct subsets Ω_i for $i \in \{1, \ldots, r\}$, s.t., all flags $\mathscr{F} \in \Omega_1 \times \cdots \times \Omega_r$ are maximal flags[3], then the geometry $\mathscr{G} = (\Omega, \mathscr{I})$ is said to be of rank r and elements of Ω_i are said to be of type i

A ranked geometry can be denoted by $\mathscr{G} = (\Omega_1, \Omega_2, \ldots, \Omega_r, \mathscr{I})$.

Example 2.5. *Ranks*

- *In the Euclidean space $\boldsymbol{E}^2$ the objects of type 1 are the points (Ω_1) and the objects of type 2 are the lines (Ω_2). All maximal flags are of the same form: they consist of a line and a point on that line. Hence Euclidean space $\boldsymbol{E}^2$ is of rank $r = 2$.*

- *The discrete square can be partitioned into "points" $\Omega_1 = \{a, b, c, d\}$ (the vertices) and "lines" $\Omega_2 = \{(a, b), (b, c), (c, d), (d, a), (a, c), (b, d)\}$ (the edges and diagonals). A maximal flag, e.g, is $\mathscr{F} = \{a, (a, b)\}$. Therefore the discrete square is also of rank $r = 2$.*

All geometries of rank $r \geq 2$ can be formulated in terms of rank $r = 2$ geometries and therefore the geometries that are of interest within this thesis are of rank $r = 2$ (see p. 4 and 5 in [Beu98]). These two types of objects will be called *points* $\mathscr{P} = \Omega_1$ and *lines* $\mathscr{L} = \Omega_2$ (or hyperplanes $\mathscr{H}$); these geometries are also called *incidence structures*[4].

From now on the points $\mathscr{P}$ are considered to be some set while the lines $\mathscr{L}$ (or hyperplanes $\mathscr{H}$) are a subset of the power set $2^{\mathscr{P}}$ of the point set $\mathscr{P}$ and the incidence relation $\mathscr{I}$ is defined in order to resemble the above mentioned "set theoretical containedness"[5]:

Theorem 2.6. *Containedness Incidence Relation $\mathscr{I}_{\text{set}}$*

Let $\mathbb{P}$ be a set, $\mathscr{H} \in 2^{\mathscr{P}}$ then $\mathscr{G} = (\mathscr{P}, \mathscr{H}, \mathscr{I}_{\text{set}})$ is a geometry for the set theoretical containedness incidence relation:

$$\mathscr{I}_{\text{set}} = \{(x, x) | x \in \mathscr{P} \cup \mathscr{H}\} \cup \{(p, h), (h, p) | p \in \mathscr{P}, h \in \mathscr{H}, p \in h\}. \tag{2.1.1}$$

[3] To be more precise: all *sets* consisting of the members of the *tupels* in $\Omega_1 \times \cdots \times \Omega_r$ are maximal flags.

[4] Wherein the two types are often called points $\mathscr{P}$ and blocks $\mathscr{B}$.

[5] By far most of the technical work done within this master's thesis is formulated using the notions of analytic geometry. In this framework it is common to use lower case letters for points p and not upper case letters P as usually within the framework of synthetic geometry. But to keep all notions within this master's thesis in a similar fashion only lower case letters will denote both points and lines.

This incidence relation encodes *containedness* in a symmetric manner: the first set ensures reflexivity and the second set includes exactly those pairs of hyperplanes and points, for which it holds true that the point is set theoretically contained in the hyperplane (or line) in a symmetric manner. Thus it really is an incidence relation. In the following the symbol $\mathscr{I}$ will be used for $\mathscr{I}_{\text{set}}$ in oder to lighten the notation.

The cardinality of the point set is by definition inherited by the geometry; in particular:

Definition 2.7. *Finite Geometry (see p. 24 in [Beu98])*

A geometry $\mathscr{G} = (\mathscr{P}, \mathscr{H}, \mathscr{I})$ *is said to be* <u>*finite*</u> *if and only if the point set* $\mathscr{P}$ *is finite.*

The Axioms

Using this framework first the more intuitive affine plane and subsequently the projective plane will be constructed by imposing *axioms* on a geometry. Finally the projective plane will be generalized to a projective space.

The idea behind both affine and projective planes is that there is always a line connecting two points (this will be called axiom **AP1**). Alone from this and the containedness incidence relation defined above (2.6) it follows that a line *cannot* intersect another line in *more than one* point. Hence *all* lines either intersect in *one* point (this is exactly axiom **P2** leading to a projective plane) or lines intersect under certain conditions (e.g., allowing for parallels: their existence and uniqueness is exactly expressed by axiom **A2** leading the affine plane). Supplementing the axioms **AP1** and **A2** with a nondegeneracy condition while indirectly encoding the "planeness" in an abstract manner results in the following axioms:

Definition 2.8. *Affine Plane (see p. 27 in [Beu98])*

An <u>*affine plane*</u> *is an incidence structure* $\mathscr{G} = (\mathscr{P}, \mathscr{L}, \mathscr{I})$ *that obeys the following axioms:*

AP1 $\forall p, q \in \mathscr{P}, p \neq q : \exists^{=1} \ell \in \mathscr{L} : p \mathscr{I} \ell \wedge q \mathscr{I} \ell$
(There is exactly one joint line for all pairs of different points.),

A2 $\forall \ell \in \mathscr{L} : \forall p \in \mathscr{P} : \neg(p \mathscr{I} \ell) \Rightarrow (\exists^{=1} m \neq \ell \in \mathscr{L} : m \cap \ell = \varnothing)$
(Exactly one parallel line exists for each pair of a line and a given point that is not incident with the line.), and

A3 $|\mathscr{L}| > 1$ *and* $\forall \ell \in \mathscr{L} : \exists^{>1} p \in \mathscr{P} : p \mathscr{I} \ell$
(There are at least two lines and at least two points per line.).

Example 2.9. *Affine Planes*

- *Once again the Euclidean plane* $\boldsymbol{E}^2$ *is paradigmatic for the notion introduced: it is an affine plane. In fact it is probably* <u>*the*</u> *inspiration for the abstract axiomatic affine plane.*

- *The discrete square is also an affine plane. E.g., the line* $\{a, b\}$ *and the point* c *have the unique parallel* $\{c, d\}$*. It is important to understand that the pretended intersection of the two diagonal is no intersection point. Both diagonals are parallel!*

Exchanging the parallel axiom **A2** with one that ensures a conceptual symmetry between points and lines (and adapting the non-degeneracy axiom **A3** while keeping the property of unique joint lines **AP1**) results in another synthetic geometry wherein all lines intersect:

Definition 2.10. *Projective Plane (see p. 5 et seq. in [Beu98])*

A <u>*projective plane*</u> *is an incidence structure* $(\mathscr{P}, \mathscr{L}, \mathscr{I})$ *that obeys the following axioms:*

AP1 $\forall p, q \in \mathscr{P}, p \neq q : \exists^{=1} \ell \in \mathscr{L} : p \mathscr{I} \ell \wedge q \mathscr{I} \ell$
(There is exactly one joint line for all pairs of different points.),

P2 $\forall \ell, m \in \mathscr{L}, \ell \neq m : \exists^{=1} p \in \mathscr{P} : \ell \mathscr{I} p \wedge m \mathscr{I} p$
(There is exactly one intersection of two different lines,), and

P3 $|\mathscr{L}| > 1$ *and* $\forall \ell \in \mathscr{L} : \exists^{>2} p \in \mathscr{P} : p \mathscr{I} \ell$
(There are at least two lines and at least three points per line).

Example 2.11. *Projective Planes*

- *A smooth two dimensional manifold that is homeomorphic to the real 2-sphere* S^2 *cut into half where opposite points of the remaining boundary are identified results in the real projective plane. The fundamental graph (where the remaining open edges have to be identified according to the direction of the arrows) is shown in Figure 2.2(a).*

- *The smallest*[6] *non-degenerate finite projective plane is called Fano plane*[7] $PG(2, 2)$*. It is the projective counterpart to the discrete square. Acutally one way to look at it is seeing it as the projective extension (or closure) of the discrete square where a line at infinity has been added, s.t., all parallels of the square do intersect, (see Figure 2.2(b)).*

[6] "Small" in terms of the number of elements.
[7] After Gino Fano (* 1871, † 1952), an italian mathematician.

If such a projective plane is finite, it is always possible to count the number of points on a line, say N, and call $q := N - 1$ the *order* of the finite geometry (such that there are $q + 1$ points on a line).

One refers to finite projective planes of order q with the symbol $PG(2, q)$. Higher ranked geometries are denoted by $PG(r, q)$, where r is the rank.

Theorem 2.12. *Number of Elements in a Projective Plane (see p. 24 et seq. in [Beu98])*

For all finite projective planes $(\mathscr{P}, \mathscr{L}, \mathscr{I})$ there exists a natural number $q \in \mathbb{N}, q \geq 2$, the order of the projective plane, s.t., there are

- $|\mathscr{P}| = q^2 + q + 1$ *points in the plane,*
- $|\mathscr{L}| = q^2 + q + 1$ *lines in the plane,*
- $q + 1$ *lines incident with a single point, and*
- $q + 1$ *points incident with a single line.*

The basal difference of both types of geometries thus lies in the difference of the axioms **A2** and **P2**.

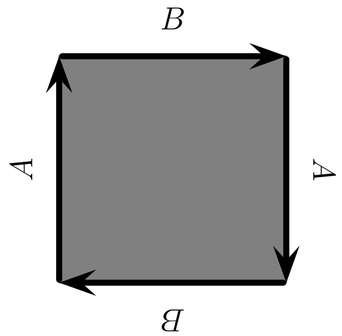
Example 2.13. *Order of and Counts in Projective Planes*

- *The Fano plane from the examples above possess 7 points and lines each and obviously a $q = 2$ exists, s.t., $2^2 + 2 + 1 = 7$. Hence there should be $2 + 1 = 3$ lines through each point and 3 points on each line, which also turns out to be true.*
- *For all prime numbers p (and even p^s with $s \in \mathbb{N}\backslash\{0\}$) there is projective plane with rank $q = p$ (or $q = p^s$ respectively). They will be coordinatized over the so-called Galois Fields and form the essential finite projective geometries of this work.*

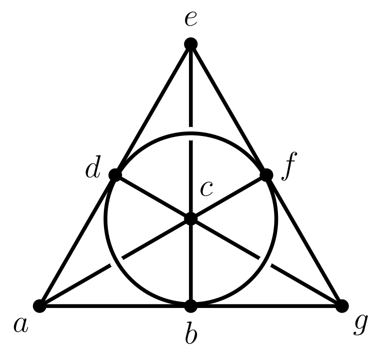
The *joint line* of two distinct points $p, q \in \mathscr{P}$ (that always exists according to axiom **A1**) is denoted by $\overline{pq} \in \mathscr{L}$ and the *intersection point* of two lines $\ell, m \in \mathscr{L}$ is denoted by $\ell \cap m \in \mathscr{P}$.

If one does not want to restrict the geometry to have the planarity that is implemented so far, one has to modify the axiom **P2**. This more general version is called Veblen-Young(-Pasch) axiom[8] and encodes that if a line intersects two sides of a triangle it also does so with the third. It restricts the application of axiom **P2** to all lines in "planes" but allows for non-intersecting skew lines in general.

[8] After Oswald Veblen (* 1880, † 1960), and American mathematician, John Wesley Young (* 1879, † 1932), also an American mathematician, and Moritz Pasch (* 1843, † 1930), a German mathematician.



(a) The real projective plane as its fundamental polygon. Opposite arrows have to be “glued” together pointing in the same direction. The resulting two-dimensional manifold cannot be embedded in $\mathbb{R}^3$ and it is not orientable.



(b) The Fano plane $PG(2, 2)$ is the smallest non-degenerate finite geometry of rank $r = 2$. As for the discrete square the lines are just symbolic to express that the corresponding points *are* the line.

Figure 2.2: The real projective plane and the Fano plane are examples of projective geometries as defined in definition 2.10 and described in example 2.11.

Definition 2.14. *Projective Space (see p. 5 et seq. in [Beu98])*

A projective space is an incidence structure $(\mathscr{P}, \mathscr{H}, \mathscr{I})$ that obeys the following axioms:

A1 $\forall p, q \in \mathscr{P}, p \neq q : \exists^{=1} \ell \in \mathscr{L} : p \mathscr{I} \ell \wedge q \mathscr{I} \ell$
(There is exactly one joint line for all pairs of different points.),

VYP $\forall p, q, r, s \in \mathscr{P} : (\exists a \in \mathscr{P} : a = \overline{pq} \cap \overline{rs} \Rightarrow \exists b \in \mathscr{P} : b = \overline{pr} \cap \overline{qs})$
(There is exactly one intersection point of two different lines in case the lines lie planar in a certain sense.), and

P3 $|\mathscr{L}| > 1$ *and* $\forall \ell \in \mathscr{L} : \exists^{>2} p \in \mathscr{P} : p \mathscr{I} \ell$
(There are at least two lines and at least three points per line.).

Example 2.15. *Veblen-Young(-Pasch) Axiom*

- *An illustration of the **VYP** axiom is the Fano plane shown in Figure 2.2(b). E.g., the line $\overline{af}$ is intersecting the line $\overline{dg}$ in the point c. To obey the **VYP** axiom the line $\overline{ac}$ and $\overline{fg}$ should meet: and they do so in e. One can easily check that this holds true for all selections of four different points. Hence there are no skew lines and not only **VYP** is fullfiled, also **P2** is. Thus the Fano plane is a projective space and in particular a projective plane.*
- *A symbolic illustration is given in Figure 2.3.*

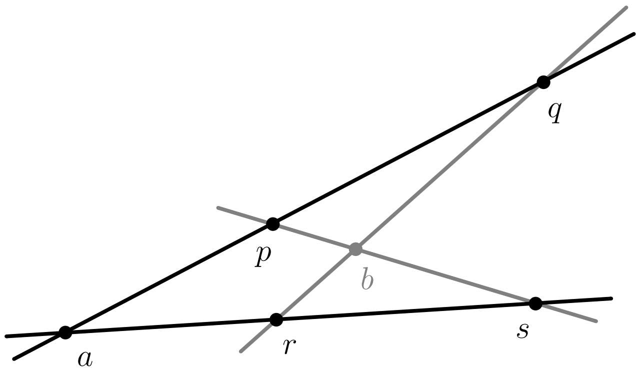


Figure 2.3: The Veblen-Young(-Pasch) axiom **VYP** stipulates that if the joint line $\overline{pq}$ of two points p and q intersects another joint line $\overline{rs}$ of two points r and s in the point a, then the two joint lines $\overline{ps}$ and $\overline{qr}$ have to intersect in a point b.

Duality

It is a lot easier to work with projective spaces instead of affine spaces, while all the notions and problems of affine geometry exist and can be formulated in projective geometry. The main reason for the simplicity in projective planes is the *duality* of points and lines *by construction*:

Definition 2.16. *Dual Geometry* $\mathscr{G}^*$

For a geometry $\mathscr{G} = (\Omega_1, \Omega_2, \mathscr{I})$ *of rank* 2 *the* <u>*dual geometry*</u> $\mathscr{G}^* = (\Omega_1^*, \Omega_2^*, \mathscr{I}^*)$ *is defined to have the same incidence relation as the geometry* $\mathscr{G}$ $(\mathscr{I}^* = \mathscr{I})$ *but the objects of type* 1 *are to be defined as the objects of type* 2 *of the geometry* $\mathscr{G}$ $(\Omega_1^* = \Omega_2)$ *and vice versa for the objects of type* 2 $(\Omega_2^* = \Omega_1)$.

In terms of lines and points this simply means:

$$\mathscr{G}^* = (\mathscr{P}^*, \mathscr{L}^*, \mathscr{I}^*) = (\mathscr{L}, \mathscr{P}, \mathscr{I}). \tag{2.1.2}$$

Theorem 2.17. *Duality of Projective Planes (see p.* 29 *et seq. in [RG09] and p.* 8 *et seq. in [Beu98])*

$\mathscr{G}$ *is a projective plane* $\Leftrightarrow$ $\mathscr{G}^*$ *is a projective plane. All theorems that are proven to be true for projective planes are as well true in the dual geometry under exchange of dual structures (e.g., points with lines and later quadrics with dual quadrics).*

This "symmetry" can be seen in the very construction of the projective plane: for two points there is always a unique joint line and for two lines there is always a unique intersection point. Later there will be quadrics that possess dual quadrics. For higher ranked projective geometries there is a duality between the 0-dimensional points and the $(d-1)$-dimensional hyperplanes.

This duality is not to be confused with an *isomorphism* in general. This circumstance is very similar — and indeed connected[9] — to the Riesz theorem identifying a vector space with its dual vector space. For finite dimensional spaces — and these are the cases we are interested in within this framework — there exists a natural isomorphism, but in case of infinite dimensional spaces neither a geometry and its dual geometry nor a vector space and its dual vector space have to be isomorphic[10].

Linear and Quadratic Sets

Two notions that will be very important for further investigations will be introduced: linear and quadratic sets. The first will turn out to be determinable by linear and the latter by quadratic equations when the geometries are formulated in an analtyic manner (using coordinates).

Definition 2.18. *Linear Set $\mathscr{U}$ (see p.* 10 *et seq. in [Beu98])*

A subset of the point set $\mathscr{U} \subset \mathscr{P}$ is called <u>*linear*</u> *if and only if for all two (distinct) points $p, q \in \mathscr{U}$ all points $r \in \mathscr{P}$ that are incident with the joint line $\overline{pq} \in \mathscr{L}$ $(p\mathscr{I})$ are also contained in this point set $\mathscr{U}$; formally:*

$$\forall p, q \in \mathscr{U} \subset \mathscr{P} : \forall r \in \mathscr{P} : r\mathscr{I}\overline{pq} \Rightarrow r \in \mathscr{U}. \tag{2.1.3}$$

A set $\mathscr{C} \subset \mathscr{P}$ is called <u>*collinear*</u> *if and only if all points $p \in \mathscr{C}$ are incident with the very same common line:*

$$\exists \ell \in \mathscr{L} : \forall p \in \mathscr{C} \subset \mathscr{P} : p\mathscr{I}\ell. \tag{2.1.4}$$

In order to define a *quadratic* set the synthetic notion of a *tangent* (and that of a tangent space at a point) is neccessary (or at least handy):

Definition 2.19. *Tangent of a Set and Tangent Space of a Point in a Set (see p.* 137 *et seq. in [Beu98])*

For a set $\mathscr{S} \subset \mathscr{P}$ and a point $p \in \mathscr{S}$, the following notions are defined:

- *A* <u>*tangent*</u> *is a line that is either* <u>*completely included*</u> *in a set or has only* <u>*one point in common*</u> *with the set.*
- *A* <u>*tangent space*</u> *at a point is the* <u>*set of tangents*</u> *to the set the point is in.*

This leads to the notion of a *quadratic set*:

[9] This connection will become clear later in terms of the analyticly coordinatized geometries.

[10] The "finite" in "finite dimension" is not to be confused with the "finite" in "finite spaces". Once it refers to the cardinality of the set of elements and once to the number basis vectors. But the finite spaces we are interested in have both finitely many elements and can be coordinatized such that they are finitely generated.

Definition 2.20. *Quadratic Set $\mathscr{Q}$ (see p. 137 et seq. in [Beu98])*

A quadratic set is a set obeying the the following two properties:

- *If three or more points of a line are included in the set $\mathscr{Q}$ all of the points are included.*
- *The tangent spaces of all points in $\mathscr{Q}$ are either linear sets or the whole point set $\mathscr{P}$.*

This intuitively encodes the closest deviation from linear structures.

Link of Projective Planes and Affine Planes

There is very deep link between affine and projective geometries. All affine geometries can be extended by adding a “line at infinity” (or a hyperplane in general) and be retrieved by removing this “points at infinity”.

Definition 2.21. *Parallellity Equivalence Relation*

*Axiom **A2** guarantees that there is always a unique parallel line ℓ' for a given line ℓ through a point p that is not incident with ℓ. If one defines a line to be parallel to itself, parallellity establishes an equivalence relation $\sim_{||} \subset \mathscr{L} \times \mathscr{L}$ defined as:*

$$\ell \sim_{||} \ell' \quad :\Leftrightarrow \quad \ell || \ell' \text{ according to } \boldsymbol{A2} \text{ or } \ell = \ell'.$$

Theorem 2.22. *Link of Projective and Affine Planes*[11] *(see p. 113 et seq. in [RG09] and p. 27 et seq. in [Beu98])*

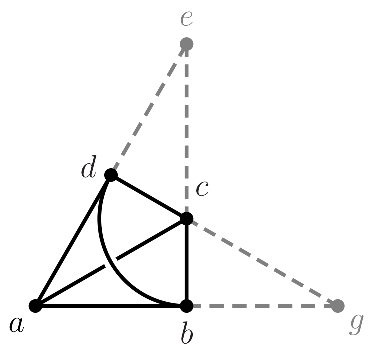
Excluding an arbitrary line, denoted by l_∞ and called “line at infinity”, from the set of lines $\mathscr{L}$ of an projective plane $(\mathscr{P}, \mathscr{L}, \mathscr{I})$ results in a new set of lines $\mathscr{L}'$ and removing the corresponding points from the point set $\mathscr{P}$ — and thus from all remaining lines — results also in a new point set $\mathscr{P}'$. This procedure leads to an affine plane $(\mathscr{P}', \mathscr{L}', \mathscr{I}')$.

Vice versa adding the points of a line that is not included in an affine plane $(\mathscr{P}', \mathscr{L}', \mathscr{I}')$ to the point set $\mathscr{P}'$ results in a new point set $\mathscr{P}$ and the corresponding line to the line set $\mathscr{L}'$ leads to a new line set $\mathscr{L}$. The resulting geometry is a projective plane $(\mathscr{P}, \mathscr{L}, \mathscr{I})$. Formally:

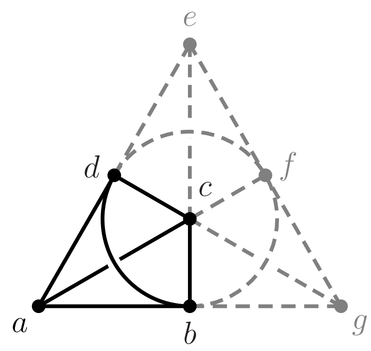
$$\forall \ell_\infty \in \mathscr{L} : (\mathscr{P}, \mathscr{L}, \mathscr{I}) \text{ projective plane} \quad \Leftrightarrow \quad (\mathscr{P}', \mathscr{L}', \mathscr{I}') \text{ affine plane}, \qquad (2.1.5)$$

where $\mathscr{L}' = \mathscr{L} \backslash \ell_\infty$, $\mathscr{P}' = \mathscr{P} \backslash \{p_\infty \in \mathscr{P} | p_\infty \mathscr{I} \ell_\infty\}$, and $\mathscr{I}' = \mathscr{I} \cap (\mathscr{P}' \times \mathscr{L}') \times (\mathscr{P}' \times \mathscr{L}')$ is the inherited incidence relation.

[11]The same link exists for projective and affine spaces in general and shall here just be examplified.



(a) In the lower left corner there is exactly the discrete square from figure 2.1(b) (up to a homeomorphism of the embedding in the plane of this very sheet it is displayed on) leading to two intersection points *at infinity* e and g...



(b) ...that define the *line at infinity* $\ell_\infty = \overline{eg}$ that contains also the intersection point f of the two parralel diagonals of the square affine plane $\overline{ac}$ and $\overline{bd}$ (displayed as the circle). The result is the Fano plane $PG(2,2)$ from Figure 2.2(b).

Figure 2.4: The process of extending an affine plane (here the discrete square) to a projective plane (the Fano plane).

Thus according to Theorem 2.12 there are p^2 points in such a finite affine plane and p points on a line respectively.

Example 2.23. *Extension of Affine to Projective Planes*

- *The affine discrete square from Fig. 2.1(b) (the smallest affine geometry in terms of the number of elements) can be extended by such a line at infinity* ℓ_∞ *and the result is exactly the Fano plane (the smallest projective geometry). See Figure 2.4(b) for an illustration.*
- *The real plane* $\mathbb{R}^2$ *extended by new intersection points of all families of parallels results in the real projective plane.*

Pappos' and Desargue's Theorems (Properties)

Up to now the geometries have been introduced merely by axioms and no *coordinates* are available in order to refer to the individual elements of the geometries. In order to do so two properties of geometries are indicators whether and how it is possible to coordinatize these geometries: Pappos' and Desargue's property. These are theorems that can be proven for real affine and projective geometries, but not in general for synthetic geometries (see p. 55 et seq. in [Beu98]). The coordinatization will be made precise in the next section, while in this part of this section both properties will be presented.

Definition 2.24. *Pappos' Property (see p. 31 in [RG09])*

A projective geometry $(\mathscr{P}, \mathscr{L}, \mathscr{I})$ is said to be Pappossian if and only if for all points $p_c \in \mathscr{P}$ it holds true that for all pairs of two different lines $\ell, \ell' \in \mathscr{L}$ that meet in p_c ($\ell \mathscr{I} p_c$ and $\ell' \mathscr{I} p_c$) one can select three different points per line (a_1, a_2, a_3 on ℓ and b_1, b_2, b_3 on ℓ') that are also different from p_c, s.t., the three intersections $c_{ij} := \overline{a_i b_j} \cap \overline{a_j b_i}$ with $i < j \in \{1, 2, 3\}$ lie on the same line l_c.

Pappos' property is illustrated in Fig. 2.5(a).

Definition 2.25. *Desargue's Property (p. 78 et seq. in [Beu98])*

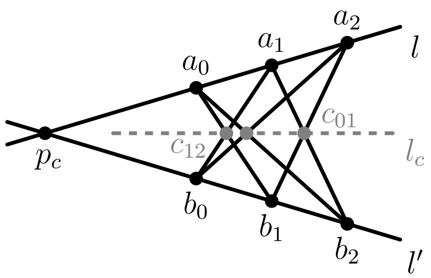
A projective geometry $(\mathscr{P}, \mathscr{L}, \mathscr{I})$ is said to be Desarguesian if and only if for all mutually different points $p_c, a_1, a_2, a_3, b_1, b_2, b_3 \in \mathscr{P}$ it holds true that, in case neither three points out of $\{p_c, a_1, a_2, a_3\}$ nor three points out of $\{p_c, b_1, b_2, b_3\}$ constitute linear sets and finally all three lines $\overline{a_i b_i}$ for $i \in \{1, 2, 3\}$ meet in p_c, all points $c_{ij} := \overline{a_i b_j} \cap \overline{a_j b_i}$ with $i < j \in \{1, 2, 3\}$ lie on a line ℓ_c.

An illustration of Desargue's property is shown in Fig. 2.5(b). Both properties can also be formulated for the corresponding affine planes, where "intersection at infinity" translates to "parallellity".

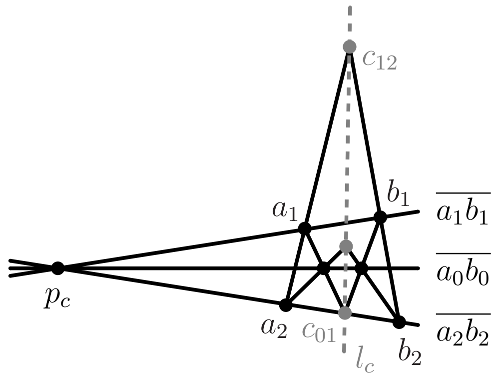
Theorem 2.26. *Facts on Pappossian and Desarguesian Geometries (see p. 59 et seq. in [Beu98] and p. 31 and 32 in [RG09])*

- *All Pappossian geometries are Desarguesian, while the inverse does not hold true in general. Hence Desargue's property is the weaker condition. This has been proven by Hessenberg (see p. 65 et seq. in [Beu98]).*
- *All Desarguesian geometries can be coordinatized over skew fields and all Papossian geometries over fields.*
- *For dimensions $d \geq 3$ all geometries are Desarguesian.*
- *All planes embedded in geometries of dimension $d \geq 3$ are Desarguesian planes.*

Within this master thesis one of the main objects of interest had been finite projective planes. But due to Theorem 2.26 and the final goal of the whole project — to look at finite projective geometries that are (most likely) four dimensional and utilitze them as the underlying geometry of physics — it is clear that all finite projective planes of interest are Desarguesian and hence all of them are coordinatizable over skew fields. It has been proven by Wedderburn [MW05] that finite skew fields are already fields. Hence the analytic algebraic structure to employ in order to coordinatize the geometries that we are interested in is a finite field.



(a) Illustration of the Pappos property (2.24). The label of the point c_{02} is ommited for clarity.



(b) Illustration of the Desargue property (2.25). The labels points a_0, b_0, and c_{02} are also ommited for clarity.

Figure 2.5: Both Pappos' and Desargue's property are fulfilled in the real affine and projective plane and therefore they can serve as an illustration of this rather abstract properties.

2.1.2 The Analytic Way to Geometries: Projective Spaces over Fields

Basic Algebraic Structures

The geometries so far lack a systematic and functional labeling of its elements. They are geometries because they fulfill certain (synthetic) axioms; but in analytic geometry one wants to manipulate the points and lines more directly: one wants to *calculate* with the elements and hence *algebraic* structures need to be employed. In order to coordinatize geometries the notions of rings, skew fields, and especially fields are neccessary. Even more, vector spaces and algebras are the structures from what the finite geometries we are interested in are constructed. All of these more advanced structures are based on the notion of a *monoid* and a *group*:

Definition 2.27. *Monoid* $(M, \circ)$ *and (Commutative) Group* $(G, \circ)$ *(see p.* 339 *and* 340 *in [Bro08])*

A monoid is a pair $(M, \circ)$ *of a set* M *and a binary group operation* $\circ : M \times M \to M$ *on the set*[12]*, s.t., the operation is associative and there is a left-neutral element. These properties are defined as follows:*

- *Associativity:* $\forall m_0, m_1, m_2 \in M : m_0 \circ (m_1 \circ m_2) = (m_0 \circ m_1) \circ m_2$
  (The order of evaluating the group operation does not matter.) and
- *Existence of a Left-neutral Element:* $\exists e \in M : \forall m \in M : e \circ m = m$
  (There is an element that has no effect on all elements when acting on them from the left.).

[12] I.e., an endomorphism of the set.

In case the operation guarantees the existence of a left-inverse element for all elements of the set the wohle structure is called a group and is denoted by $(G, \circ)$*. This property is defined more precisely as:*

- *Existence of a Left-Inverse Element:* $\forall g \in G : \exists g^{-1} \in G : g^{-1} \circ g = e$
  (For all elements in the set there is an element, s.t., the latter acting on the former yields the neutral element of the underlying monoid.)

If the group (or only the monoid) obeys commutativity the structure is called commutative (or Abelian), where this is defined as:

- *Commutativity:* $\forall m_0, m_1 \in M : m_0 \circ m_1 = m_1 \circ m_0$
  (The order of how two arbitrary elements are operating on each other does not matter.)

Example 2.28. *Monoids and Groups*

- $(\mathbb{Z}, +)$*: The integer numbers* $\mathbb{Z}$ *are in combination with the usual plus operation* $+$ *a commutative group with the neutral element* 0*.*
- $(\mathbb{N}, \cdot)$*: the natural numbers* $\mathbb{N}$ *equipped with the multiplication are "only" a monoid with the neutral element* 1*.*

Working with groups reveals that it might be handy to execute the group (or monoid) operation several times to calculate within this structure. This successive execution of the operation might be abstracted as another operation (like the well-known multiplication of integers that is inherited from the successive execution of additions). But an additional group operation does not have to be induced by the "original" operation. In general there could be just two different operations acting on the set underlying a group or monoid leading to the question whether the set in combination with this additional operation behaves again like a monoid, group or "even" a commutative group. There are several combinations possible of how a structure with respect to one operation can be paired with structures over the same set with respect to another operation:

Definition 2.29. *Ring* $(R, +, \circ)$ *and (Skew) Field* $(\mathbb{F}, +, \cdot)$ *(see p. 363 et seq. in [Bro08])*

Let $(R, +)$ *be a commutative group and* $(R, \circ)$ *be a monoid. The tripel* $(R, +, \circ)$ *is called a ring, if the two operations* $+$ *and* $\circ$ *are left-distributive:*

- $(R, +, \circ)$ *Ring* $:\Leftrightarrow$ $(R, +)$ *commutative Group,* $(R, \circ)$ *monoid, and*
- *Left-distributivity:* $\forall r_0, r_1, r_2 \in R : r_0 \circ (r_1 + r_2) = r_0 \circ r_1 + r_0 \circ r_2$
  (The "factor" is distributed symmetrical over the two "summand".).

In case the commutative group with respect to one operation is not joined by a monoid with respect to the other, but instead by a group the resulting structure is called a skew fields. More precisely:

- $(\mathbb{F}, +, \circ)$ *Skew Field* $:\Leftrightarrow$ $(\mathbb{F}, +, \circ)$ *ring,* $(\mathbb{F}\backslash\{e_+\}, \circ)$ *group.*

If $(\mathbb{F}, \circ)$ *is also commutative the whole structure is called a field and "$\circ$" is often denoted by "$\cdot$":*

- $(\mathbb{F}, +, \cdot)$ *Field* $:\Leftrightarrow$ $(\mathbb{F}, +, \cdot)$ *ring,* $(\mathbb{F}\backslash\{e_+\}, \cdot)$ *commutative group.*

For finite sets underlying skew fields Wedderburn [MW05] showed — as already mentioned — that if left-distributivity is supplemented by right-distributivity the strucutre is already a field, because then the "multiplication" is automatically commutative.

The examples for these structures will follow after the next definitions related to the *congruence* (modulus) equivalence relation, because on the one hand structures defined in their context serve as nice examples of rings and (skew) fields, while on the other hand the yet mentioned — but not yet defined — finite Galois fields are the fields over which the projective geometries we are interested in are coordinatized. Galois fields are crucially depending on the notion of the modulus equivalence relation.

Definition 2.30. *Equivalence Relation* $\sim$ *(see p. 337 in [Bro08])*

Let G *be a set and* $\sim \subset G \times G$ *be a relation. If* $\sim$ *fulfills the properties of an incidence relation (symmetry and reflexivity) and is also transitive it is called an equivalence relation. This additional property is defined as follows:*

- *Transitivity:* $\forall g_0, g_1, g_2 \in G : g_0 \sim g_1 \wedge g_1 \sim g_2 \Rightarrow g_0 \sim g_2$
  (The equivalence can be transfered via a common element.).

Definition 2.31. *Equivalence Classes* $[g]_\sim$ *(see p. 337 in [Bro08])*

Let G *be a set and* $\sim$ *be an equivalence realtion over* G*. All elements that are equivalent to a given element* g *in terms of the equivalence relation* $\sim$ *are grouped into a set called equivalence class* $[g]_\sim$*. The whole set* G *can be partitioned into disjunct equivalence classes. One element of an equivalence class is named (and can serve) as a representant:*

$$[g]_\sim :\Leftrightarrow \{g' \in G | g' \sim g\}. \tag{2.1.6}$$

The brackets of equivalence classes are often ommited and one representant is used as if it was the whole class.

Definition 2.32. *Congruence Equivalence Relation* $\sim_{\text{mod } m}$ *(see p. 337 in [Bro08])*

For a ring $(R, +, \circ)$ *the following relation is an equivalence relation for all* $m \in R\backslash\{e_\circ\}$*:*

$$\forall r, r' \in R : r \sim_{\text{mod } m} r' :\Leftrightarrow \exists n \in R\backslash\{e_\circ\} : r - r' = n \circ m. \tag{2.1.7}$$

m *is called the modulus of this equivalence relation. One denotes the equivalence also by* $r = r' \text{mod } m$ *or simply if the modulus* m *is clear* $r \equiv r'$ *(or even* $r = r'$*).*

The question arises whether the equivalenve classes equipped with the two inherited operations of the original ring omit also a ring or even a field structure.

In case the original ring is $(\mathbb{Z}, +, \cdot)$ the set of all equivalence classes is denoted by $\mathbb{Z}/m\mathbb{Z}$ and it is isomorph to $\{0, 1, \ldots, m-1\} \subset \mathbb{N}$ with both operations $+$ and $\cdot$ yielding the remainder of their original result for a division by m.

Example 2.33. *Rings and (Skew) Fields*

- $(\mathbb{Z}, +, \cdot)$: *The integers together with the usual addition and multiplication behave like a ring.*
- $(\mathbb{Z}/m\mathbb{Z}, +_{mod\ m}, \cdot_{mod\ m})$: *The integers from* 0 *up to* $m-1$ $(m > 0)$ *are together with the addition and multiplication to the modulus of* m *a ring.*
- $(\mathbb{R}, +, \cdot)$: *The real numbers* $\mathbb{R}$ *together with the addition and multiplication are a field.*
- $(\mathbb{Z}/p\mathbb{Z}, +_{mod\ p}, \cdot_{mod\ p}, p\ prim) =: \mathbb{F}_p$: *Congruence rings to the modulus of a prime number* p *are fields.*

The last example is a special case of the so called *Galois fields* $\mathbb{F}_q$. They are not restricted to prime p-many numbers in the underlying set, but allow $q = p^s$-many elements (with $s \in \mathbb{N}\backslash\{0\}$). They are the fields that coordinatize finite projective geometries. But within this work $s = 1$, because for higher values of s the multiplication is not the usual integer multiplication to the modulus of q, but must be constructed by extending the field $\mathbb{F}_p$ to $\mathbb{F}_{p^s}$ similarly as, e.g., the complex numbers $\mathbb{C}$ to the quaternios $\mathbb{H}$. Furthermore there is no obvious reason, why we should look at them[13].

All finite fields with $q = p^s$ elements are unique in the sense that they are isomorhpic to $\mathbb{F}_q$ as fields (see p. 85 et seq. in [Bos06]).

More Advanced Algebraic Structure: Vector Spaces and Algebras

Definition 2.34. *Left R-Module* $(M, R, +_M, \circ_{RM})$ *and* $\mathbb{F}$*-Vector Space* $(V, \mathbb{F}, +_V, \circ_{\mathbb{F}V})$ *(see p. 367 in [Bro08])*

A R-module $(M, R, +, \circ_{RM})$ *is a quadrupel of a set* M*, a ring* R *and two binary operations* $+_M : M \times M \to M$ *and* $\circ_{RM} : R \times M \to M$*, s.t.,* $(M, +_M)$ *is a commutative group that fulfills in combination with the ring* R *the additional properties*

- *Associativity of* $\circ_{RM}$: $\forall r, r' : \forall m \in M : r \circ_{RM} (r' \circ_{RM} m) = (r \cdot_R r') \circ_{RM} m$
  (It does not matter whether one first multiplies two ring elements and then multiplies the resulting element with a module element or one executes the two $\circ_{RM}$ *mulitplications successively.)*

[13] On the opposite there is no obvious reason, why not to look at them and hence one should keep them in mind, because simplicity might be a flawed guiding principle!

If the ring R is also a field, the whole structure is called a vector space $(V, \mathbb{F}, +_V, \circ_{\mathbb{F}V})$:

- $(V, \mathbb{F}, +_V, \circ_{\mathbb{F}V})$ *vector space* $:\Leftrightarrow$ $(V, +_V, \circ_{\mathbb{F}V})$ *left $\mathbb{F}$-module, $\mathbb{F}$ field.*

Theorem 2.35. *Dual Vector Space V^* (see p. 370 et seq. in [Kna13])*

Let V be a vector space over a field $\mathbb{F}$. The quadrupel $(V^*, \mathbb{F}, +^*, \circ^*)$ is a vector space, called the dual vector space V^* of V for

- *V^* is set of all linear maps ϕ from V to $\mathbb{F}$, where linear means:* $\forall f \in \mathbb{F}, v, w \in V : \phi(v +_V f \cdot_{\mathbb{F}V} w) = \phi(v) +_V f \cdot_{\mathbb{F}} \phi(w)$,
- $\forall v \in V, \phi, \psi \in V^* : (+^* : V^* \times V^* \to V^*$ *by* $(\phi + \psi) := \phi(v) +_{\mathbb{F}} \psi(v))$, *and*
- $\forall f \in \mathbb{F}, v \in V, \phi \in V^* : (\circ^* : \mathbb{F} \times V^*$ *by* $(f \circ^* \phi)(v) := f \cdot_{\mathbb{F}} \phi(v))$.

Usually the quadrupel is presented as a tripel leaving out the explicit denotion of the ring R (or the field $\mathbb{F}$) or only as the set M, for a module, and V for a vector space. Up to now there is only an addition that closes as a binary operation in the modules and vector spaces. If this operation is joined by a second binary operation that closes in M or V that works nicely together with the already existing operations, namely it is left- and right-distributive and compatible with the multiplication of ring elements with module elements $\circ_{RM}$ and the ring multiplication $\circ_R$ (or analogously for fields with $\circ_{\mathbb{F}V}$ and $\circ_{\mathbb{F}}$) it is called an (abstract) algebra (over a ring or field respectively)[14]:

Definition 2.36. *Algebra Over a Ring $(M, \bullet_M)$ or Field $(V, \bullet_V)$*

$\bullet_M$ is said to be compatible with a left R-module structure if and only if the following property is satisfied:

- *Compatibility of $\bullet_M$ with $\circ_{RM}$ and $\circ_R$* $:\Leftrightarrow \forall r, r' \in R : \forall m, m' \in M :$
  $(r \circ_{RM} m) \bullet_M (r' \circ_{RM} m') = (r \circ_R r') \circ_{RM} (m \bullet_M m')$.
  (The $\bullet_M$ operation is R-linear in both entries.).

Algebras can now be contructed both over modules and vector spaces:

- $(M, \bullet_M)$ *Algebra Over a Module M* $:\Leftrightarrow$ *M module, $\bullet_M$ left- and right-distributive, and compatible $\circ_{RM}$ and $\circ_R$,*
- $(V, \bullet_V)$ *Algebra Over a Vector Space V* $:\Leftrightarrow$ *V vector space, $\bullet_V$ left- and right-distributive, and compatible with $\circ_{\mathbb{F}M}$ and $\circ_{\mathbb{F}}$.*

Example 2.37. *Modules, Vector Spaces and Algebras*

- $(R^n, +_R, \cdot_{\mathbb{N}R})$*: For any ring R the cartesian product R^n can be equipped with the component-wise $\cdot_{\mathbb{N}R}$ multiplication with natural numbers and the component wise R-addition $+_R$ resulting in a new ring,*

[14] These conditions encode *R-bilinearity.*

- $(\mathbb{F}_p^n, +_{\mathbb{F}_p}, \cdot_{\mathbb{F}_p\mathbb{F}_p^n})$: *The n-fold cartesian product of the Galois fields $\mathbb{F}_p$ equipped with the component-wise multiplication with a Galois scalar (an element of the Galois field $\mathbb{F}_p$) and the component-wise $\mathbb{F}_p$-multiplication yields a vector space, and*

- $(\mathbb{R}^3, +_{\mathbb{R}}, \cdot_{\mathbb{R}\mathbb{R}^3}), (\mathbb{R}^3, \times_{\mathbb{R}^3})$: *The real three-dimensional vector space $\mathbb{R}^3$ equipped with the component-wise $\mathbb{R}$-addition and -multiplication is a $\mathbb{R}$-vector space that equipped with the cross-product $\times_{\mathbb{R}^3}$ is an abstract algebra. The lines, points, and planes constitute an example of a coordinatized affine geometry.*

The notions developed so far encode all that is needed to coordinatize affine and projective geometries — as well finite as infinite geometries.

2.1.3 Coordinatization of Finite Projective and Affine Spaces

For the construction of a d-dimensional ($d \in \mathbb{N}\backslash\{0\}$) Desarguesian projective space over a field $\mathbb{F}$ one needs the notion of *homogeneous* coordinates.

Definition 2.38. *Homogenity Equivalence Relation $\sim$ and Homogenous Coodinates (see p. 47 and 48 in [RG11])*

The homogenity equivalence relation $\sim$ over a vector space build from the $(d+1)$-fold Cartesian product of the field $\mathbb{F}^{d+1}$ is defined by imposing that for an element $p \in \mathbb{F}^{d+1}\backslash\{0\}$ the element $\lambda \cdot p$ with $\lambda \in \mathbb{F}\backslash\{0\}$ does lie in the same equivalence class as p:

$$\forall\ p, p' \in \mathbb{F}^{d+1}\backslash\{0\} :\ p \sim p' \quad :\Leftrightarrow \quad \exists \lambda \in \mathbb{F}\backslash\{0\} : p = \lambda \cdot p'. \tag{2.1.8}$$

The homogenous coordinates are defined to be the equivalence classes of this equivalence relation.

Finally this in turn leads to the definition of the d-dimensional projective space:

Definition 2.39. *Projective space $\mathbb{P}^d\mathbb{F}$*

$$\mathbb{P}^d\mathbb{F} := {}^{\mathbb{F}^{d+1}}/_{\sim} = \{[p]_{\sim} |\ p \in \mathbb{F}^n\backslash\{0\}\}. \tag{2.1.9}$$

That this is indeed a geometry and in particular a projective space will become clear later in terms of the incidence relation defined in Definition 2.42 (see Theorem 2.43). All the zero dimensional points $p \in \mathbb{P}^d\mathbb{F}$ could be grouped in disjunct sets P_i for

$i \in \{0, \dots, d\}$ with one component normed to e — the neutral element of $(\mathbb{F}, +)$ — due to homogenity each[15]:

$$\mathbb{P}^d\mathbb{F} =: P = \bigcup_{i=0}^{d} P_i \quad \text{with}$$
$$P_i = \left\{ \begin{pmatrix} p_1 \\ \vdots \\ p_i \\ e \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; p_1, \dots, p_i \in \mathbb{F} \right\} \quad \text{and} \tag{2.1.10}$$
$$\forall i, j \in \{0, \dots, d\} : P_i \cap P_j = \emptyset$$

and the cardinalities are thus given by

$$\forall i \in \{0, \dots, d\} : |P_i| = |\mathbb{F}|^i \quad \text{and hence}$$
$$|P| = \sum_{i=0}^{d} |P_i| = \sum_{i=0}^{d} |\mathbb{F}|^i = \frac{1 - |\mathbb{F}|^{d+1}}{1 - |\mathbb{F}|}. \tag{2.1.11}$$

For two dimensions this recovers the result from Theorem 2.12. The $(d-1)$-dimensional hyperplanes h lie embedded as $(d-1)$-dimensional linear subspaces in the projective space itself. But due to the well-known theorem by Riesz (see p. 372 in [Kna13]) the vector space and its dual space are isomorphic for a finite dimension. Hence the hyperplanes can be identified with the elements in the dual projective space $\mathbb{P}^d\mathbb{F}^*$ equivalently defined using the dual space $(\mathbb{F}^{d+1})^*$ of the vector space leading to the very same form of hyperplanes as for the points:

Definition 2.40. *Dual Projective Space* $\mathbb{P}^d\mathbb{F}^*$

$$(\mathbb{P}^d\mathbb{F})^* := {}^{(\mathbb{F}^{d+1})^*}/_{\sim} = \{[h]_\sim | \; h \in (\mathbb{F}^{d+1})^* \backslash \{0\}\}. \tag{2.1.12}$$

All the hyperplanes $h \in \mathbb{P}^d\mathbb{F}^*$ could be grouped like it has been done for the points

[15]Thus the elements of this d-dimensional space are represented by vectors with $(d+1)$-many components up to a scalar factor. In the following a multiplication by a non-zero scalar factor does not have to be mentioned if the object is coordinatized using homogenous coordinates.

as:

$$\mathbb{P}^d\mathbb{F}^* =: H = \bigcup_{i=0}^{d} H_i \quad \text{with}$$

$$H_i = \left\{ \begin{pmatrix} h_1 \\ \vdots \\ h_i \\ e \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; h_1, \dots, h_i \in \mathbb{F} \right\} \quad \text{and} \tag{2.1.13}$$

$$\forall i, j \in \{0, \dots, d\} : H_i \cap H_j = \emptyset$$

and the cardinalities are equal as in the case of the points.

For finite dimensions the two spaces are isomorph ($\mathbb{P}^d\mathbb{F}^* \cong \mathbb{P}^d\mathbb{F}$) due to the similar definition process and the isomorphism between $\mathbb{F}^{d+1}$ and $(\mathbb{F}^{d+1})^*$. Hence geometric duality holds true for points and hyperplanes.

Example 2.41. *Coordinatized Projective Geometries*

- *The projective planes* $\mathbb{P}^2\mathbb{F}$ *(rank* 2 *and fulfilling* ***P2****) are exactly coordinatized as* $(\mathbb{P}^2\mathbb{F}, \mathbb{P}^2\mathbb{F}^*, \mathscr{I})$ *with points* $P = \{(f, f', e)^t | f, f' \in \mathbb{F}\} \cup \{(f, e, 0) | f \in \mathbb{F}\} \cup \{(e, 0, 0)\} = \mathbb{P}^2\mathbb{F}$ *and the lines* $L = \{(f, f', e)^t | f, f' \in \mathbb{F}\} \cup \{(f, e, 0) | f \in \mathbb{F}\} \cup \{(e, 0, 0)\} = \mathbb{P}^2\mathbb{F}^*$.
- $PG(2,2) = \mathbb{P}^2\mathbb{F}_2$*: The Fano plane* $PG(2,2)$ *from example (2.4(b)) is coordinatized over the finite field* $\mathbb{F}_2 = \{0, 1\}$ *yielding the points parameterized as* $a = (0,0,1)^t$, $b = (0,1,1)^t$, $c = (1,1,1)^t$, $d = (1,0,1)^t$, $e = (1,0,0)^t$, $f = (1,1,0)^t$, $g = (0,1,0)^t$ *and the lines analogously.*

In order to check whether this structures obey the axioms of projective or affine geometries (Definitions 2.8, 2.10) one is in need of an incidence relation. One could utilize the set theoretical incidence relation as defined in 2.6; but it is possible (and very handy) to transform the condition to be included in this incidence relation into a linear algebraic equation. Therefore a new definition is neccessary:

Definition 2.42. *Dot Product* $\circ$ *of a Point* $p \in \mathbb{P}^d\mathbb{F}$ *and a Hyperplane* $h \in \mathbb{P}^d\mathbb{F}^*$

$$\circ : \mathbb{P}^d\mathbb{F} \times \mathbb{P}^d\mathbb{F}^* \to \mathbb{F}, \tag{2.1.14}$$

$$(p, h) \mapsto p \circ h := \sum_{i=1}^{d+1} h^i \cdot p_i (=: p^t h =: h^t p =: h^i p_i). \tag{2.1.15}$$

The last definition sign implements the usage of the Einstein sum convention.

The coordinatized projective spaces defined above are — despite the name — only algebraic structures and no geometries. But the yet defined dot product contains all information that is needed to construct the set theoretical containedness incidence relation in terms of a simple vanishing condition. A point inserted in the Hesse form of a hyperplane equals zero if and only if the point lies in the hyperplane and hence the dot product of the point p and the hyperplane h (or line l) have to vanish:

Theorem 2.43. *Dot Product and Set Theoretical Containedness Incidence Relation*

$$\mathscr{I}_\circ := \{(p,h) \in P \times H \mid p \circ h = 0\} \cap \{(g,g) \in P \times H\} \tag{2.1.16}$$

is equivalent to the above defined set theoretical containedness incidence relation $\mathscr{I}_{\text{set}}$ *from Definition 2.6 and* $(\mathbb{P}^d\mathbb{F}, \mathscr{I}_\circ)$ *fulfills the axioms of a rank d Desarguesian projective geometry, while* $(\mathbb{F}^d, \mathscr{I}_\circ)$ *fulfills the axioms of a d-dimensional Desarguesian affine space.*

In particular the two dimensional cases fulfill also the axioms for projective and affine planes.

The geometrical objects are the linear subspaces of the underlying vector spaces. Hence the rank r of a d-dimensional geometry is really d, because there are d-many types of linear subspaces: points (0-dimensional), lines (1-dimensional), planes (3-dimensional), ..., $(d-1)$-dimensional hyperplanes.

It is useful to define a generalized cross-product in both $\mathbb{P}^d\mathbb{F}$ and $\mathbb{P}^d\mathbb{F}^*$:

Definition 2.44. *Cross-Product of Points in* $\mathbb{P}^d\mathbb{F}$ *or Hyperplanes in* $\mathbb{P}^d\mathbb{F}^*$

For both $v_1 = p_1, \dots, v_d = p_d \in \mathbb{P}^d\mathbb{F}$ *or* $v_1 = h_1, \dots, v_d = h_d \in \mathbb{P}^d\mathbb{F}^*$ *the cross-product is defined as:*

$$\begin{aligned} (\mathbb{P}^d\mathbb{F} \times \ldots \times \mathbb{P}^d\mathbb{F}) &\to \mathbb{P}^d\mathbb{F}^* \quad \textit{or} \\ ((\mathbb{P}^d\mathbb{F})^* \times \ldots \times (\mathbb{P}^d\mathbb{F})^*) &\to \mathbb{P}^d\mathbb{F}, \end{aligned}$$

$$(v_1, \dots, v_d) \mapsto X(v_1, \dots, v_d) := \det \begin{pmatrix} — & v_1 & — \\ & \vdots & \\ — & v_d & — \\ \hat{e}_1 & \dots & \hat{e}_{d+1} \end{pmatrix} \tag{2.1.17}$$

with $\hat{e}_i$ being the base vectors of $\mathbb{P}^d\mathbb{F}$ or $(\mathbb{P}^d\mathbb{F})^*$ for $i \in \{1, \dots, d+1\}$. The determinant has to be seen as a formal operations here.

Example 2.45. *Cross-Product in Two Dimensions*

The "normal" cross-product of vectors with three components is retrieved as follows:

$$\begin{aligned}\vec{a} \times \vec{b} &= det \begin{pmatrix} — & \vec{a}^t & — \\ — & \vec{b}^t & — \\ \hat{e}_1 & \hat{e}_2 & \hat{e}_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \end{pmatrix} = \\ &= a_1 b_2 \hat{e}_3 + a_2 b_3 \hat{e}_1 + a_3 b_1 \hat{e}_2 - a_3 b_2 \hat{e}_1 - a_1 b_3 \hat{e}_2 - a_2 b_3 \hat{e}_1 = \\ &= (a_2 b_3 - a_3 b_2)\hat{e}_1 + (a_3 b_1 - a_1 b_3)\hat{e}_2 + (a_1 b_2 - a_2 b_1)\hat{e}_3 = \\ &= \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}. \end{aligned} \tag{2.1.18}$$

The following consideration is the inspiration for the notion of a *joint* structure: the cross-product is anti-symmetric in all its entries due to the anti-symmetry of the determinant under exchange of rows (or columns), while the dot product is symmetric. Hence according to the incidence $\mathscr{I}_\circ$ relation a point lies in a hyperplane of which it is a factor in a (generalized) cross-product:

$$\begin{aligned} &\mathrm{X}(h_1, \ldots, h_d) = p, \quad \mathrm{X}(p_1, \ldots, p_d) = h \\ &\forall i \in \{1, \ldots, d\} : h_i.p = 0, p_i.h = 0. \end{aligned} \tag{2.1.19}$$

Definition 2.46. *Joint Objects (Joints)*

- *Joint hyperplane h of the points $p_1, \ldots, p_d$: $h := X(p_1, \ldots, p_d)$.*

- *Joint point (intersection) p of $h_1, \ldots, h_d$: $p := X(h_1, \ldots, h_d)$.*

Example 2.47. *Joints in Two Dimensions*

For $d = 2$ and $\mathbb{F} = \mathbb{F}_3$, the joint line $\overline{p_1 p_2}$ of the two points

$$p_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } p_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{2.1.20}$$

is given by their "normal" cross-product

$$\overline{p_1 p_2} = p_1 \times p_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \tag{2.1.21}$$

where $-1 = 2$ (for $\mathbb{F}_3$) had been used. Both points should lie on this line, which can easily be tested by evaluating the dot product of the points with the line in order to *check their incidence:*

$$p_1^t\overline{p_1p_2} = \begin{pmatrix}1 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix} = 3 = 0 \text{ and}$$
$$p_2^t\overline{p_1p_2} = \begin{pmatrix}0 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix} = 3 = 0. \tag{2.1.22}$$

Explicite Form of Point and Hyperplane Sets and Their Cardinalities for $\mathbb{F}_q$

For the Galois fields $\mathbb{F}_q$ the cardinality is simply $|\mathbb{F}_q| = q$ and the neutral element with respect to the multiplication is simply the 1. Hence the resulting *types* of the projective spaces are $\mathbb{P}^d\mathbb{F}_q =: P$ and $\mathbb{P}^d\mathbb{F}_q^* =: H$.

H and P together with the operations on the sets shall be called *finite projective geometry* $(P, L, \mathscr{I}_\circ)$. The previous results can be expressed using the new notation in the following way:

$$\begin{aligned}\mathbb{P}^d\mathbb{F}_q =: P &= \bigcup_{i=0}^{d} P_i \quad \text{with}\\ P_i &= \left\{\begin{pmatrix}p_1\\ \vdots\\ p_i\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix},\ p_1,\dots,p_i \in \mathbb{F}_q\right\} \quad \text{and}\\ \forall i,j &\in \{0,\dots,d\} : P_i \cap P_j = \emptyset\end{aligned} \tag{2.1.23}$$

and the cardinalities are given by

$$\begin{aligned}\forall i \in \{0,\dots,d\} : |P_i| = q^i \quad \text{and hence}\\ |P| = \sum_{i=0}^{d} |P_i| = \sum_{i=0}^{d} q^i = \frac{1-q^{d+1}}{1-q}\end{aligned} \tag{2.1.24}$$

and similar for the hyperplanes (lines).

Now it is possible to present an explicit form of the set of all points being incident with an arbitrary hyperplane and vice versa.

Theorem 2.48. *Incident Obejects*

For all $h \in H$ it holds true that for a h that is in H_i all the points that lie in h are given by:

$$P_i^{in}(h) := \left(\bigcup_{k=i+1}^{d} P_{ki}^{\circ}\right) \cup \left(\bigcup_{j=2}^{i+1} P_{ji}^{*}(h)\right) \quad \textit{with}$$

$$P_{ki}^{\circ} = \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda_{i+2} \\ \vdots \\ \lambda_k \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \mid \lambda_{i+2}, \ldots, \lambda_k \in \mathbb{F}_p \right\}, \tag{2.1.25}$$

$$P_{ji}^{*}(h) = \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \lambda_j \\ \vdots \\ \lambda_i \\ -(h_{j-1} + \sum_{k=j}^{i} h_k \lambda_k) \\ \lambda_{i+2} \\ \vdots \\ \lambda_{d+1} \end{pmatrix} \mid \begin{matrix} \lambda_m \in \mathbb{F}_p, \\ m \in \{j, \ldots, i, \\ i+2, \ldots, d+1\} \end{matrix} \right\}, \tag{2.1.26}$$

$$\forall k, k' \in \{1, \ldots, d\}, j, j' \in \{2, \ldots, d\} :$$
$$P_{ki}^{\circ}(h) \cap P_{k'i}^{\circ}(h) = \emptyset, P_{ji}^{*}(h) \cap P_{j'i}^{*}(h) = \emptyset, P_{ki}^{\circ}(h) \cap P_{ji}^{*}(h) = \emptyset,$$
$$\textit{thus } |P_{ki}^{\circ}| = q^{k-(i+1)} \textit{and } |P_{ji}^{*}(h)| = q^{d+1-j}.$$

This rather complicated form arises from a case-by-case analysis: either the upper i possibly non-vanishing components of h are only multiplied with zeros while all remaining components of h are itself zeros that get multiplied with the remaining components of the point (these are the P_{ki}°) or it might happen that non-vanishing components of the two objects "see" each other when the dot product is evaluated. In this case the sum of all such values has to cancel, what is achieved by the negative term in the $P^*_{ji}(h)$. Splitting all incident objects in these classes leads easily to the cardinality of $P_i^{\text{in}}(h_i)$:

$$
\begin{aligned}
|P_i^{\text{in}}(h)| &= \sum_{k=i+1}^{d} |P^\circ_{ki}| + \sum_{j=2}^{i+1} |P^*_{ji}(h_i)| = \\
&= \sum_{k=i+1}^{d} q^{k-(i+1)} + \sum_{j=2}^{i+1} q^{d+1-j} = \\
&= \sum_{k=0}^{d-i-1} q^k + \sum_{k=d-i}^{d-1} q^k = \\
&= \sum_{k=0}^{d-1} q^k = \frac{1-q^d}{1-q}.
\end{aligned}
\tag{2.1.27}
$$

Each $h \in H$ is element of exactly one of the H_i and all the points in $P_i^{\text{in}}(h)$ are all the points that are incident with h:

$$\forall h \in H : \exists^{=1} H_i : (h \in H_i \wedge \forall p \in P_i^{\text{in}}(h) : p \circ h = 0). \tag{2.1.28}$$

Therefore one has the complete set of points incident with a given hyperplane h[16]. By duality one can interchange H with P and p with h resulting in the explicitely parameterized form of all hyperplanes through a given point.

Example 2.49. *Points on a Hyperplane at Infinity h_∞*

- *A hyperplane at infinity h_∞ is just one of the hyperplanes in H, because no hyperplane is distinguished. But an especially important form of a hyperplane for later considerations is the following (with $\overrightarrow{h}_\infty \in \mathbb{F}_p^d$): $h_\infty = (\overrightarrow{h}^t_\infty, 1)^t$. Thus instrumentalizing the apparatus constructed above the points $p_\infty \in P_d^{\text{in}}(h_\infty)$ that are incident with hyperplanes of the form $h_\infty = (\overrightarrow{h}^t_\infty, 1)^t$ have this form (with $\overrightarrow{p_\infty} \in \mathbb{F}_p^d$): $p_\infty = (\overrightarrow{p}^t_\infty, -\overrightarrow{h}^t \overrightarrow{p}_\infty)^t$. These are all the P^*_{jd} for all j from 1 up to d, while all the components of the points in P°_d vanish completely and hence are no valid points*[17]*. In this case it is obvious that the dot product yields zero:*

$$h^t_\infty p_\infty = \begin{pmatrix} \overrightarrow{h}^t_\infty & 1 \end{pmatrix} \begin{pmatrix} \overrightarrow{p}_\infty \\ -\overrightarrow{h}^t_\infty \overrightarrow{p}_\infty \end{pmatrix} = \overrightarrow{h}^t_\infty \overrightarrow{p}_\infty - \overrightarrow{h}^t_\infty \overrightarrow{p}_\infty = 0 \Leftrightarrow p_\infty \mathscr{I}_\circ h_\infty. \tag{2.1.29}$$

- *The points in the corresponding affine plane (all remaining points) must have a non-vanishing dot product with this hyperplane $h_\infty = (\overrightarrow{h}^t_\infty, 1)^t$. Thus they are of the form: $p_{\text{aff}} = (\overrightarrow{p}^t_{\text{aff}}, 1 - \overrightarrow{h}^t \overrightarrow{p}_{\text{aff}})^t$ for an arbitrary $\overrightarrow{p}_{\text{aff}} \in \mathbb{F}_p^d$.*

---

[16] One has to define $\bigcup_{i=a}^{b} S_i := \emptyset$ for $b < a$ (for arbitrary sets S_i), in order to deal with degeneracies that might occur in the unification.

[17] Thus one also has to exclude all formally occuring vectors that have only vanishing components, because they are no legal homogenous coordinates.

For the rest of this work — if not stated different explicetely — the underlying field of all coordinatized geometries is a Galois field $\mathbb{F}_p$ $(s = 1)$ corresponding to projective geometries $PG(d,p) = \mathbb{P}^d\mathbb{F}_p$, wherein the points and the hyperplanes are the dual geometrical objects.

All other *linear geometrical objects* are given by affine-linear combinations (the spans[18] up to homogenity) of i-many points, if i is the dimension of the subspace:

Definition 2.50. *Geometric Objects of Type i in Projective Geometries of Rank d*

The objects u_i of type i are exactly linear combinations (up to homogenity) of $i+1$ points (of which no three different points are collinear). The set of all object of type i $\mathscr{U}_i$ (Ω_i) is defined as follows:

$$\mathscr{U}_i = \{u_i \in \mathscr{P} \mid u_i = \mathrm{span}_{\mathbb{F}}(p_0, \ldots, p_i) \text{ for } p_0, \ldots, p_i \in \mathscr{P}\}.$$

Example 2.51. *Lines ℓ in Projective Geometries of Rank d*

For all lines $\ell \in \mathscr{L}$ there are at least two points on it, say p and p' and the line is their span: $\ell = \mathrm{span}_{\mathbb{F}}(p,p') = \{p'' \in \mathscr{P} \mid p'' = f \cdot p + f' \cdot p'$ with $k, k' \in \mathbb{F}\}$, which is due to homogenity equivalent to $\ell = \mathrm{span}_{\mathbb{F}}(p,p') = \{p'' \in \mathscr{P} \mid p'' = p + f' \cdot p'$ with $k' \in \mathbb{F}\} \cup \{p\}$, the affine-linear combination, where the point p has to be added seperately, because homogenity does not account for the possible zero factor. In this form it is obvious that there are $p+1$ points on a line in all finite projective spaces of an arbitrary dimension and thus not only in projective planes as stated above in 2.12.

Example 2.52. *A Coordinatized Finite Projective Plane $\mathbb{P}^2\mathbb{F}_3$ The next successor in size of the Fano plane is the finite projective plane $\mathbb{P}^2\mathbb{F}_3$. It contains 13 points and 13 lines. It can be seen as the result of the extension of a finite affine plane with 9 points and three families of parallels. Its coordinatization as well as all families of parallels and the line at infinity are shown in Figure 2.6.*

[18] The $\mathrm{span}_{\mathbb{F}}$ is defined as the set of all $\mathbb{F}$-linear combination of its arguemnts: $\mathrm{span}_{\mathbb{F}}(v_0, \ldots, v_n) = \{v \in R \mid \exists f_0, \ldots, f_n \in \mathbb{F} : v = \sum_{k=0}^{n} f_k \cdot v_k\}$ for a vector space V.

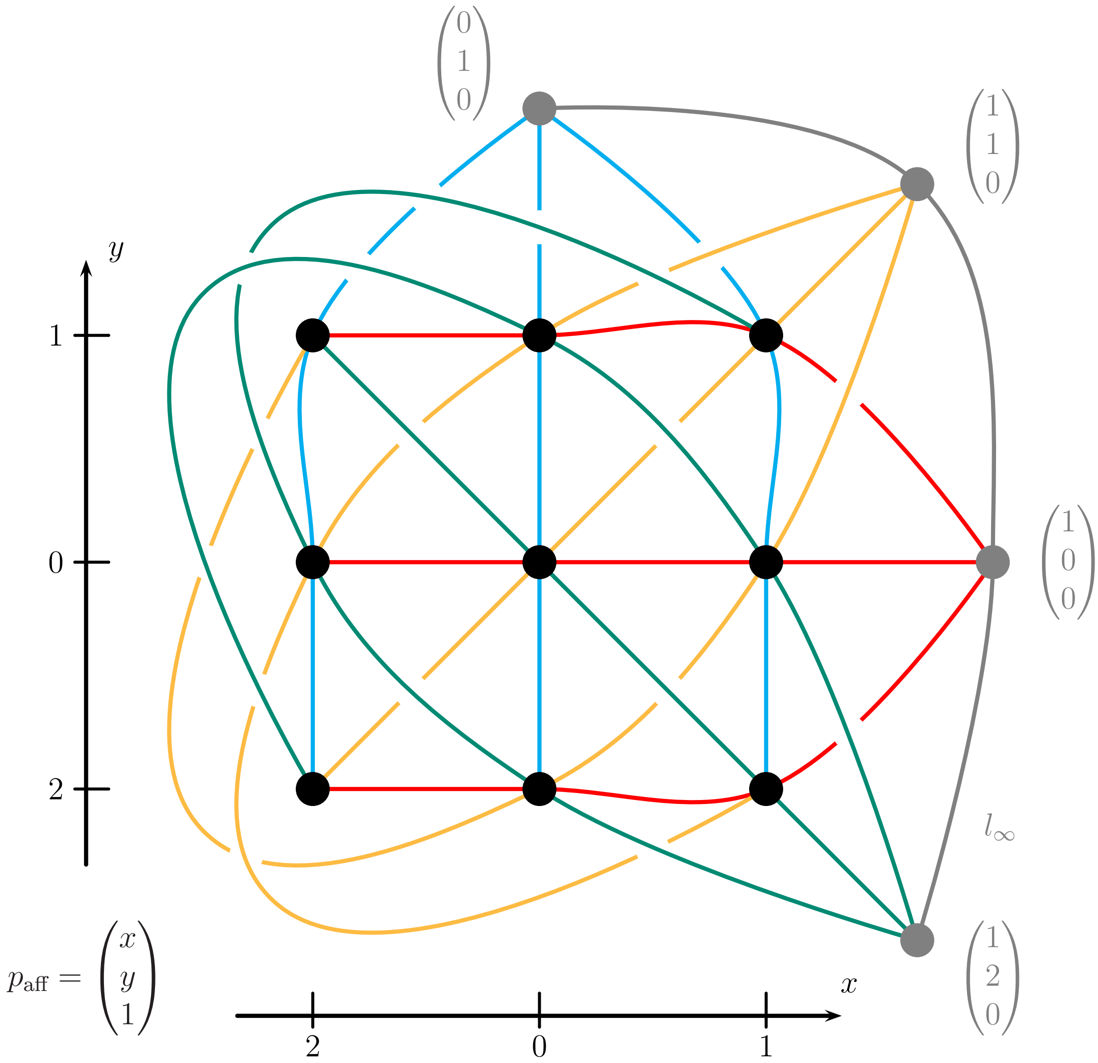


Figure 2.6: The second smallest finite projective plane $PG(2,3) = \mathbb{P}^2\mathbb{F}_3$ with all points and lines. It is structured according to the particular choice of a line at infinity: $l_\infty = (0,0,1)^t$. The resulting $p^2 = 3^2 = 9$ affine points p_{aff} are black. The $p+1 = 4$ points p_∞ on the line at infinity are gray. All lines that share a color belong to the same family of parallels: they meet in the same point at infinity.

2.1.4 Quadrics in Projective Spaces

According to the incidence relation given by the dot product defined in 2.42, the points p on a hyperplane h (that form a *linear set*) obey a *linear equation*: $p \circ h = \sum_{i=1}^{d} p_i h^i = 0$. They are the kernel of the *linear form* given by the components of the hyperplane. But the synthetic *quadratic sets* do not obey linear equations. Some of them obey quadratic equations: the so called *quadrics*. They are simply the kernel of *quadratic forms*:

Definition 2.53. *Quadratic Form q*

A quadratic form[19] is a map $q : V \to \mathbb{F}$ from a vector space $(V, \mathbb{F}, +_V, \circ_{\mathbb{F}V})$ onto its underlying field $\mathbb{F}$ obeying the following two properties:

- *Quadratic Factorizing: $\forall v \in V : \forall f \in \mathbb{F} : q(f \circ_{\mathbb{F}V} v) = f^2 \circ_{\mathbb{F}} q(v)$.*
  (Factors pull out squared.),
- *Symmetric and Bilinear Polarform B: The map $B : V \times V \to \mathbb{F}$ defined as $B(v, w) := q(v + w) - q(v) - q(w)$ is both symmetric[20] and bilinear[21].*

This implicit definition is not useful in order to actually write down a quadratic form and execute calculations with it. For vector spaces there is a matrix respresentation of bilinear forms and hence of quadrics. The quadrics we are interested in are quadrics of finite projective spaces $\mathbb{P}^d\mathbb{F}_p$:

Theorem 2.54. *Matrix Representation of Projective Quadratic Forms*

All projective quadratic forms $q : \mathbb{P}^d\mathbb{F}_p \to \mathbb{F}_p$ can equivalently be expressed in terms of a matrix in $\mathbb{P}^{d\times d}\mathbb{F}$ (determined up to homogenity):

$$\forall q \text{ quadratic form on } \mathbb{P}^d\mathbb{F}_p : \exists M \in \mathbb{P}^{d\times d}\mathbb{F}_p : \forall p \in \mathbb{P}^d\mathbb{F}_p : q(p) = p^t M p. \tag{2.1.30}$$

Definition 2.55. *Projective Quadric Q_M*

For a matrix $M \in \mathbb{P}^{d\times d}\mathbb{F}_p$ the quadric Q_M is the kernel of the quadratic form belonging to that matrix:

$$Q_M := \{p \in P | q(p) = p^t M p = M_{ij} p^i p^j = 0\}. \tag{2.1.31}$$

The matrix M can be seen as a $^0_2$-tensor[22] that maps a point $p \in P = \mathbb{P}^d\mathbb{F}$ ($\hat{=}\, ^1_0 T(\mathbb{P}^d\mathbb{F}_p)$) onto a hyperplane ($\hat{=}\, ^0_1 T(\mathbb{P}^d\mathbb{F}_p)$), its dual hyperplane, the so-called *polar*:

[19] The symbol q is already in use for the cardinality of the fields $\mathbb{F}_q$ but quadratic forms are also usually denoted by q and hence the symbol occurs for both objects. From the context it should be clear what the meaning of q is.

[20] Symmetry: $\forall v, w \in V : B(v, w) = B(w, v)$.

[21] Bilinearity: Due to the symmetry it suffices to stipulate linearity in one of the arguments: $\forall f \in \mathbb{F}, \forall a, b, w \in V : B(a +_V f \circ_{\mathbb{F}V} b, w) = B(a, w) +_{\mathbb{F}} f \cdot_{\mathbb{F}} B(b, w)$.

[22] A ^p_q-tensor is a map that maps p elements of the dual vector space V^* and q elements of the vector space V linearly in all arguments into the underlying field $\mathbb{F}$.

Definition 2.56. *Polar Hyperplane $pol_M(p)$ of a Point p and a Quadrik Q_M*

For a quadric Q_M the polar hyperplane $pol_M(p)$ is defined as the image of the point p under the matrix $M : P \to H$:

$$\mathrm{pol}_M(p) := \mathrm{Sym}(M)p \in H, \tag{2.1.32}$$

where $\mathrm{Sym}(M)$ is the symmetric part of M. It can be calculated by: $\mathrm{Sym}(M) = 1/2(M + M^t)$. The quadric in terms of the polar is the set of all points for which it holds true that the point is incident with its polar:

$$p \in Q_M \Leftrightarrow p^t M p = p^t \mathrm{pol}_M(p) = 0. \tag{2.1.33}$$

The concept of a polar cannot only be used to decide whether a point lies *on* the quadric, it allows for a condition to decide whether a point lies *inside* or *outside* a quadric:

Definition 2.57. *Relative Location of a Point p With Respect a Quadric Q_M*

A point $p \in P$ is said to lie

- *inside* $Q_M :\Leftrightarrow |pol_M(p) \cap Q_M| = 0$,
- *on* $Q_M \Leftrightarrow |pol_M(p) \cap Q_M| = 1$, *and*
- *outside* $Q_M \Leftrightarrow |pol_M(p) \cap Q_M| > 1$.

The first condition is a definition, the second follows — as explained above — from the dot incidence relation in combination with the definition of the polar, and the third condition is a consequence of the two previous ones, because it is the only case that remains.

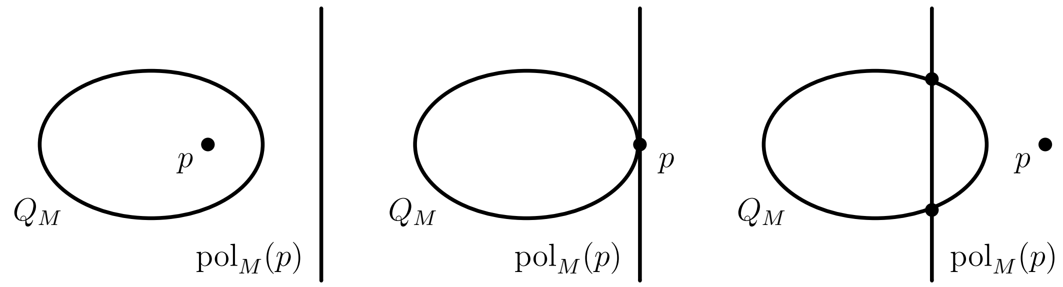
Example 2.58. *Relative Location of a Point and a Quadric in Two Dimensions*

For projective planes $\mathbb{P}^2\mathbb{F}_p$ this means that if the polar of a point intersects the quadric twice, that the point lies inside of it and if the polar does not intesect the quadric at all the point lies outside (see Figure 2.7).

Definition 2.59. *Non-Degenerate Quadric*

A quadric Q_M is called "non-degenerate" if and only if the symmetric part of the representing matrix M is of full rank[23].

[23] I.e., there is a basis transformation, s.t., the resulting diagonal matrix has dimension d-many non-vanishing entries on this diagnonal.



(a) The point p lies by definition *inside* the quadric Q_M, because the polar $\mathrm{pol}_M(p)$ has *no* point in common with Q_M.

(b) If a point p lies *on* its own polar $\mathrm{pol}_M(p)$ it satiesfies directly the condition to lie *on* the quadric Q_M itself.

(c) If the polar $\mathrm{pol}_M(p)$ intersects the quadric at least twice, the point p is defined to lie *outside* Q_M.

Figure 2.7: Whether a point p lies *inside*, *on*, or *outside* a quadric Q_m is defined according to the intersection behavior of the *polar hyperplane* $\mathrm{pol}_M(p)$ of the point. This is illustrated for the 2-dimensional case, where the polar hyperplanes are lines.

Theorem 2.60. *Quadratic Sets and Quadratic Forms (see p.* 161 *et seq. in [Beu98])*

The point set Q_M is a quadratic set in the synthetic sense defined above in Definition 2.20 if and only if the quadric is non-degenerate, i.e., the symmetric part of the matrix M is non-degenerate.

Degenerate quadrics of lower rank do not resemble quadratic sets; they resemble, e.g., the whole space for $M = 0$ (rank 0) or a line for rank 1. In the rest of this master thesis all matrices M belonging to projective quadrics are assumed to be non-degenerate. Without loss of generality the matrix M can be choosen to be symmetric, because the quadratic form is symmetric in the components of p and the anti-symmetric part of the matrix M would not contribute to the value $q(p)$; then the quadric condition can easily be split into a diagonal and off-diagonal terms: $\sum_{i=1}^{d+1} M_{ii}(p^i)^2 + \sum_{i<j=1}^{d+1} M_{ij}p^ip^j = 0$. This can be seen from the following calculation:

$$\begin{aligned}
M_{ij}p^ip^j &= (\frac{1}{2}(M_{ij} + M_{ji}) + \frac{1}{2}(M_{ij} - M_{ji}))p^ip^j = \mathrm{Sym}(M)_{ij}p^ip^j, \text{ because} \\
\frac{1}{2}(M_{ij} - M_{ji})p^ip^j &= -\frac{1}{2}(M_{ji} - M_{ij})p^ip^j = \\
&= -\frac{1}{2}(M_{ji} - M_{ij})p^jp^i = -\frac{1}{2}(M_{ij} - M_{ji})p^ip^j \text{ and thus} \\
&\Rightarrow \frac{1}{2}(M_{ij} - M_{ji})p^ip^j = 0. \qquad (2.1.34)
\end{aligned}$$

Additionally it only for a symmetric M holds true that the non-degeneracy of the matrix M goes along with the non-degeneracy of the point set Q_M in the sense that Q_M is a quadratic set and not a linear space. The polar is also only well-defined if the matrix is symmetric. Therefore all matrices that underlie quadric are considered to be symmetric and non-degenerate without loss of generality in the rest of this work.

Example 2.61. *Quadrics and Polars For example consider the following matrix and the corresponding quadratic form:*

$$M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \to \quad q_M(p) = \begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = p_1 p_2$$

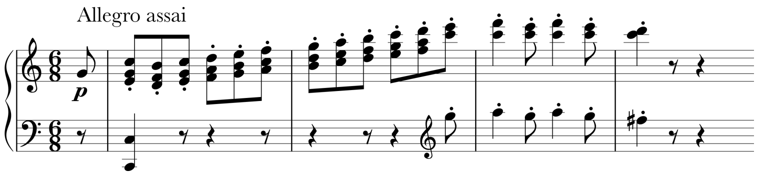
and the symmetriced matrix M_{sym} with its corresponding quadratic form:

$$M_{\text{sym}} = \frac{1}{2}(M+M^t) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \quad \to \quad q_{M_{\text{sym}}}(p) = \begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = p_1 p_2 = q(p).$$

They share the same quadratic form and thus the same points on the quadric ($q_M(p) = 0$), but $\det(M) = 0$ and $\det(M_{\text{sym}}) \neq 0$! Furthermore, the non-symmetric matrix M would lead to a different polar, if one would not have restricted the matrix in the definition of a polar 2.56 to its symmetric part:

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \textit{ but } \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}. \tag{2.1.35}$$

2.2 Automorphisms of Projective Spaces



*(*Ludwig van Beethoven, Piano Sonata in C major,
Op. 2, No. 3, Fourth Movement*)*

The name "projective geometry" arises from the fact that under projections in the geometry incidence is an invariant. But there are more automorphism of projective space than projections; all of them preserve incidence, but most of them have more structure than this plain fact. Some of their properties become obvious when the symmetry of the points and hyperplanes is broken by selecting a hyperplane at infinity and thus selecting an affine subspace of the projective space. First this process will be made precise under the names "dehomogenization" and "homogenization" and subsequently automorphisms in the affine space, called "affinities" and the embedding of these affinities as projectivities is presented, because all of that will be a tool to develop an explicitely parameterized form of biquadrics. *Translations* are a prominent representative of affinities. They will also become important in order to construct the explicetely parameterized form of biquadrics. All of the concepts are formulated explicetely in a form that enables one to directly calculate with them employing the matrix calculus and not only implicitely by their properties.

2.2.1 Projectivities

The automorphisms of a projective space, i.e., the bijective endomorphisms[24] $\mathbb{P}^d\mathbb{F}_p$, can be represented as the non-degenerate component matrices up to homogenity ($\in \mathbb{P}^{d\times d}\mathbb{F}_p$) of ${}^{1}_{1}$-tensors over the projective space and are called "*projectivities*" Π. They map points p, i.e., ${}^{1}_{0}$-tensors, onto points p', i.e., ${}^{1}_{0}$-tensors,[25] thus that they are not to be confused with the matrices M of quadrics:

Definition 2.62. *Transformation Behavior of Points p*

Let $\Pi : \mathbb{P}\mathbb{F}_q \to \mathbb{P}\mathbb{F}_q$ be a projectivity. A point $p \in P$ is then defined to transform as:

$$p' := \Pi p \quad or \quad (p')^i = \Pi^i_j p^j. \tag{2.2.36}$$

[24] Endomorphisms are maps from a set into that very same set.

[25] Points can be seen as ${}^{1}_{0}$-tensors because the natural isomorphism of a vector space and the dual of the dual vector space ($V \cong (V^*)^*$)). The whole tensor-vocabulary is introduced here to sensibilize for the formal domains of objects in order not to combine objects in a way that does not make any sense and it expresses the duality naturally.

Immediately the question arises how hyperplanes transform if points transform like in Definition 2.62, because hyperplanes are "made" by points and hence their transformation behavior should be determined by the transformation behaviour of the points *in* it:

Theorem 2.63. *Transformation Behavior of Hyperplanes h*

Let $\Pi : \mathbb{PF}_q \to \mathbb{PF}_q$ *be a projectivity transforming points as defined in Definition 2.62. Then hyperplanes* $h \in H$ *transform as:*

$$h' = \Pi^{-t} h \quad \text{or} \quad (h')_i = (\Pi^{-1})_i^j h_j. \tag{2.2.37}$$

This transformation behavior[26] preserves incidence of points and hyperplanes:

$$\begin{aligned}(p')^t h' = (\Pi p)^t (\Pi^{-t} h) = p^t \Pi^t \Pi^{-t} h = p^t \mathbb{1} h = p^t h \\ \Rightarrow \quad (p^t h = 0 \quad \Leftrightarrow \quad (p')^t h' = 0).\end{aligned} \tag{2.2.38}$$

The same holds true for quadrics: their transformation behavior is also induced by the transformation behavior of the points:

Theorem 2.64. *Transformation Behavior of Quadrics* Q_M

A quadric Q_M *(as a point set) transforms in such a way that the matrix* M' *of the transformed quadric* $Q_{M'}$ *is the result of the following transformation of* M[27]*:*

$$M' = \Pi^{-t} M \Pi^{-1} \quad \text{or} \quad M'_{ij} = (\Pi^{-1})_i^k (\Pi^{-1})_j^l M_{kl} \tag{2.2.39}$$

Proof. Incidence of points and quadrics is also preserved:

$$\begin{aligned}(p')^t M' p' = (\Pi p)^t (\Pi^{-t} M \Pi^{-1})(\Pi p) = p^t \Pi^t \Pi^{-t} M \Pi^{-1} \Pi p = p^t M p \\ \Rightarrow \quad ((p')^t M' p' = 0 \quad \Leftrightarrow \quad p^t M p = 0).\end{aligned} \tag{2.2.40}$$

□

[26] This transformation behavior can directly be derived from the "joint hyperplane calculation method" (see Definition 2.46) by the cross-product and an identity for the explicit calculation of inverses using determinants and thus cross-products (see p. 292 in [Kna13]). In two dimensions: $\ell' := p'_0 \times p'_1 = (Tp_0) \times (Tp_1) = T^{-t}(p_0 \times p_1) = T^{-t}\ell$. Thus this transformation behavior is really induced by the transformation of the points and does not have to be stipulated by the condition that incidence is preserved. This fact in turn is a consequence and can be proven without running into circular reasoning.

[27] Note that all objects transform according to how tensors transform and that the duality of points and hyperplanes corresponds to the duality of a vector space and its dual vector space.

2.2.2 Affinities

Affinities Within Affine Spaces

Affinities are collineations of affine spaces, i.e., automorphisms of affine space that map linear subspaces onto linear subspaces. Examples are rotations, translations, and dilations; but in general all affinities of an affine space $\mathbb{F}_p^d$ can be represented as affine-linear matrix transformations:

Definition 2.65. *Affinities α in Affine Spaces $\mathbb{F}_p^d$ (see p. 310 in [Kna13])*

Affinities $\alpha : \mathbb{F}^d \to \mathbb{F}^d$ are automorphisms of affine spaces $\mathbb{F}_p^d$ that map linear subspaces $\mathscr{U}$ onto linear subspaces $\mathscr{U}' = \alpha(\mathscr{U})$. For all such affinities there is a matrix $\mathbf{A} \in \mathbb{F}^{d\times d}$ and a, so-called, translation (or shift) vector $\vec{t} \in \mathbb{F}_p^d$, s.t.:

$$\alpha_{\mathbf{A},\vec{t}} : \ \vec{p} \mapsto \mathbf{A}\vec{p} + \vec{t}. \tag{2.2.41}$$

Dehomogenization and Homogenization

“Slicing” the projective geometry by selecting a hyperplane at infinity h_∞ results in an affine geometry with respect to that particular hyperplane at infinity h_∞. Within the remaining p^d points one can introduce affine coordinates $\vec{p}$[28] that are no longer homogenous.

There has to be a mapping of the homogenous coordinates of the affine points in the projective space and the affine coordinates of the very same points in an alone-standing affine-space and an inverse map that assigns to each affine coordinatized point its counterpart in the projective geometry. These maps are called *dehomogenization* $\mathcal{D}_{h_\infty} : \mathbb{P}^d\mathbb{F}_p \backslash \{h_\infty\} \to \mathbb{F}_p^d$ and *homogenization* $\mathcal{H}_{h_\infty} : \mathbb{F}_p^d \to \mathbb{P}^d\mathbb{F}_p\backslash\{h_\infty\}$ respectively. In order to find these mappings expicitely it is useful to look at the simple case of $h_\infty = (\vec{0}^t, 1)^t$. As shown in Example 2.49 the points that lie in the affine plane belonging to this hyperplane at infinity have the form $p = (\vec{p}^t, 1)^t$. Hence up to recoordinatizations within the affine plane the dehomogenization looks like this: $\mathcal{D}_{(\vec{0}^t,1)^t}(p) = \vec{p}$. But for an arbitrary h_∞ this assignment is not clear.

The homogenization in this simple case ($h_\infty = (\vec{0}^t, 1)^t$) is given by:

$$\mathcal{H}_{(\vec{0}^t,1)^t}(\vec{p}) = (\vec{p}^t, 1)^t. \tag{2.2.42}$$

[28] The vector arrow on top of the letter denotes that the vector lives in $\mathbb{F}_p^d$ and not in $\mathbb{P}^d\mathbb{F}_p$ like the projective point p without a vector arrow does.

As for the dehomogenization the generalization of the homogenization is not obvious at first hand. But in this simple case one can easily check that both maps are their respective inverses:

$$\mathcal{D}_{(\vec{0}^t,1)^t}\left(\mathcal{H}_{(\vec{0}^t,1)^t}(\vec{p})\right) = \mathcal{D}_{(\vec{0}^t,1)^t}\left(\begin{pmatrix}\vec{p}^t\\1\end{pmatrix}\right) = \vec{p} \quad \text{and}$$
$$\mathcal{H}_{(\vec{0}^t,1)^t}\left(\mathcal{D}_{(\vec{0}^t,1)^t}\left(\begin{pmatrix}\vec{p}^t\\1\end{pmatrix}\right)\right) = \mathcal{H}_{(\vec{0}^t,1)^t}(\vec{p}) = \begin{pmatrix}\vec{p}^t\\1\end{pmatrix}. \tag{2.2.43}$$

The idea that yields $\mathcal{D}_{h_\infty}$ and $\mathcal{H}_{h_\infty}$ explicitely is to utilize projectivities in order to map between the situation where the hyperplane at infinity has the easy form $h_\infty = (\vec{0}^t, 1)^t$ and the general case $(\vec{h}^t_\infty, \tilde{h}_\infty)^t$, where $\vec{h}^t_\infty \in \mathbb{F}^d_p$ and $\tilde{h}_\infty \in \mathbb{F}$. Due to the form of the hyperplane in the easy case a *block structure* of the transformation matrices is useful.

The following ansatz has been used. The projectivity $\Pi_{h_{\infty\to(\vec{0}^t,1)^t}}$ that maps $(\vec{h}^t_\infty, \tilde{h}_\infty)^t$ to $(\vec{0}^t, 1)^t$ is structured and parameterized as:

$$\Pi_{h_\infty\to(\vec{0}^t,1)^t} = \begin{pmatrix}\pi & \vec{\mu}\\ \vec{\nu}^t & \rho\end{pmatrix} \quad \text{and} \quad \hat{\Pi}^{-1}_{h_\infty\to(\vec{0}^t,1)^t} = \begin{pmatrix}\hat{\pi} & \hat{\vec{\mu}}\\ \hat{\vec{\nu}}^t & \hat{\rho}\end{pmatrix}, \tag{2.2.44}$$

where π and $\hat{\pi}$ are two matrices in $\mathbb{F}^{d\times d}$ that are not inverse to each other, because only $\Pi_{h_\infty\to(\vec{0}^t,1)^t}$ and $\hat{\Pi}^{-1}_{h_{\infty\to(\vec{0}^t1)^t}}$ are inverse to each other[29]. $\vec{\mu}$, $\hat{\vec{\mu}}$, $\vec{\nu}$, and $\hat{\vec{\nu}}$ are elements of $\mathbb{F}^d$ and ρ and $\hat{\rho}$ are scalars from $\mathbb{F}$.

From the condition that $\Pi_{h_{\infty}\to(\vec{0}^t,1)^t}$ maps $(\vec{h}^t_\infty, \tilde{h}_\infty)^t$ ($\tilde{h} \in \mathbb{F}_p$) onto $(\vec{0}^t, 1)^t$ it follows that $\vec{\nu} = \vec{h}_\infty$ and that $\rho = \tilde{h}_\infty$:

$$\hat{\Pi}^{-t}_{h_\infty\to(\vec{0}^t,1)^t}\begin{pmatrix}\vec{h}^t_\infty\\ \tilde{h}^t_\infty\end{pmatrix} \overset{!}{=} \begin{pmatrix}\vec{0}\\1\end{pmatrix} \Leftrightarrow \Pi^t_{h_\infty\to(\vec{0}^t,1)^t}\begin{pmatrix}\vec{0}\\1\end{pmatrix} = \begin{pmatrix}\pi & \vec{\nu}\\ \vec{\mu}^t & \rho\end{pmatrix}\begin{pmatrix}\vec{0}\\1\end{pmatrix} = \begin{pmatrix}\vec{\nu}\\ \rho\end{pmatrix} \overset{!}{=} \begin{pmatrix}\vec{h}^t_\infty\\ \tilde{h}^t_\infty\end{pmatrix}.$$

The two inverse conditions

$$\Pi_{h_\infty\to(\vec{0}^t,1)^t}\hat{\Pi}^{-1}_{h_\infty\to(\vec{0}^t,1)^t} = \mathbb{1} \quad \text{and} \quad \hat{\Pi}^{-1}_{h_\infty\to(\vec{0}^t,1)^t}\Pi_{h_\infty\to(\vec{0}^t,1)^t} = \mathbb{1} \tag{2.2.45}$$

yield more conditions on the components of $\Pi_{h_\infty\to(\vec{0}^t,1)^t}$ and $\hat{\Pi}^{-1}_{h_\infty\to(\vec{0}^t,1)^t}$, s.t., the remaining degrees of freedom are those of, e.g., π and $\vec{\mu}$:

$$\begin{pmatrix}\pi & \vec{\mu}\\ \vec{\nu}^t & \rho\end{pmatrix}\begin{pmatrix}\hat{\pi} & \hat{\vec{\mu}}\\ \hat{\vec{\nu}}^t & \hat{\rho}\end{pmatrix} = \left(\begin{array}{c|c}\pi\hat{\pi} + \vec{\mu}\hat{\vec{\nu}}^t & \pi\hat{\vec{\mu}} + \hat{\vec{\mu}}\rho\\ \hline \vec{\nu}^t\hat{\pi} + \rho\hat{\vec{\nu}}^t & \vec{\nu}^t\hat{\vec{\mu}} + \rho\hat{\rho}\end{array}\right) \overset{!}{=} \begin{pmatrix}\mathbb{1}_{d\times d} & \vec{0}\\ \vec{0}^t & 1\end{pmatrix}, \text{ and}$$
$$\begin{pmatrix}\hat{\pi} & \hat{\vec{\mu}}\\ \hat{\vec{\nu}}^t & \hat{\rho}\end{pmatrix}\begin{pmatrix}\pi & \vec{\mu}\\ \vec{\nu}^t & \rho\end{pmatrix} = \left(\begin{array}{c|c}\hat{\pi}\pi + \hat{\vec{\mu}}\vec{\nu}^t & \hat{\pi}\vec{\mu} + \vec{\mu}\hat{\rho}\\ \hline \hat{\vec{\nu}}^t\pi + \hat{\rho}\vec{\nu}^t & \hat{\vec{\nu}}^t\vec{\mu} + \hat{\rho}\rho\end{array}\right) \overset{!}{=} \begin{pmatrix}\mathbb{1}_{d\times d} & \vec{0}\\ \vec{0}^t & 1\end{pmatrix}, \tag{2.2.46}$$

[29]This is the reason for the additional hats on the symbols: not to confuse $\hat{\pi} \neq \pi^{-1}$.

leading to

$$\begin{aligned} \pi\hat{\pi} + \vec{\mu}\,\hat{\vec{\nu}}^t &= \mathbb{1}_{d\times d},\ \hat{\pi}\pi + \hat{\vec{\mu}}\,\vec{\nu}^t = \mathbb{1}_{d\times d}, \\ \pi\hat{\vec{\mu}} + \hat{\vec{\mu}}\rho &= \vec{0},\ \hat{\pi}\vec{\mu} + \vec{\mu}\hat{\rho} = \vec{0}, \\ \vec{\nu}^t\hat{\pi} + \rho\hat{\vec{\nu}}^t &= \vec{0}^t,\ \hat{\vec{\nu}}^t\pi + \hat{\rho}\vec{\nu}^t = \vec{0}^t, \\ \vec{\nu}^t\hat{\vec{\mu}} + \rho\hat{\rho} &= 1, \text{ and } \hat{\vec{\nu}}^t\vec{\mu} + \hat{\rho}\rho = 1. \end{aligned} \tag{2.2.47}$$

Additionally it follows from the inverse building method in terms of weighted minors (see p. 292 et seq. in [Kna13]) and homogenity:

$$\rho = \det(\hat{\pi}) \text{ and } \hat{\rho} = \det(\pi). \tag{2.2.48}$$

Thus the dehomogenization $\mathcal{D}_{h_\infty}$ and homogenization $\mathcal{H}_{h_\infty}$ with respect to an arbitrary hyperplane at infinity h_∞ are reducible to the easy case by the projective transformations $\Pi^t_{h_\infty\to(\vec{0}^t,1)^t}$ and their inverses.

Definition 2.66. *Dehomogenization $\mathcal{D}_{h_\infty}$ and Homogenization $\mathcal{H}_{h_\infty}$*

For a point $p \in \mathbb{P}^d\mathbb{F}_p$ and an affine point $\vec{p} \in \mathbb{F}_p^d$ in the affine space remaining after excluding an arbitrary $h_\infty \in \mathbb{P}^d\mathbb{F}_p^*$ from the projective geometry the two maps are:

- *Dehomogenization* $\mathcal{D}_{h_\infty}$:
  $\mathbb{P}^d\mathbb{F}_p\backslash\{p_\infty \in \mathscr{P} | p_\infty \in h_\infty\} \to \mathbb{F}_p^d$ by $\mathcal{D}_{h_\infty}(p) := \mathcal{D}_{(\vec{0}^t,1)^t}\left(\Pi_{h_{\infty\to(\vec{0}^t,1)^t}}p\right)$,
- *Homogenization* $\mathcal{H}_{h_\infty}$:
  $\mathbb{F}_p^d \to \mathbb{P}^d\mathbb{F}_p\backslash\{p_\infty \in \mathscr{P} | p_\infty \in h_\infty\}$ by $\mathcal{H}_{h_\infty}(\vec{p}) := \hat{\Pi}^{-1}_{h_\infty\to(\vec{0}^t,1)^t}\mathcal{H}_{(\vec{0}^t,1)^t}(\vec{p})$.

The dehomoginzation is not to be confused with the upper d components of the projective point p! This is only the case for the hyperplane $(\vec{0}^t, 1)^t$. But it is possible to write down an explicit *matrix form* of both maps that generalizes the "pick the upper d components"-rule of the easy case[30]. For a point p that is (nevertheless) without loss of generality structured as $p = (\vec{p}^t, \tilde{p})^t$ the explicit forms of the dehomognizations and homogenizations with respect to arbitrary hyperplanes at infinity can be calculated:

[30] Using the two conventions that $\begin{pmatrix} A \\ \vec{b}^t \end{pmatrix} \vec{x} := \begin{pmatrix} A\vec{x} \\ \vec{b}^t\vec{x} \end{pmatrix}$ and that $\begin{pmatrix} A & \vec{b} \end{pmatrix} \begin{pmatrix} \vec{x} \\ \tilde{x} \end{pmatrix} := A\vec{x} + \vec{b}\tilde{x}$. They are in total agreement to the general matrix product (see p. 181 et seq. in [Kna13]).

Theorem 2.67. *Explicite Matrix Representation of $\mathcal{D}_{h_\infty}$ and $\mathcal{H}_{h_\infty}$*

For a point $p \in \mathbb{P}^d\mathbb{F}_p \setminus \{p_\infty \in \mathscr{P} | p_\infty \in h_\infty\}$ and an affine point $\vec{p} \in \mathbb{F}_p^d$ in the affine space resulting from selecting an arbitrary hyperplane at infinity $h_\infty \in \mathbb{P}^d\mathbb{F}_p^*$ the dehomogenization $\mathcal{D}_{h_\infty}(p)$ and the homogenization $\mathcal{H}_{h_\infty}(\vec{p})$ are given by:

- *Dehomogenization* $\mathcal{D}_{h_\infty}(p) = \begin{pmatrix}\pi & \vec{\mu}\end{pmatrix} p$ *and*
- *Homogenization* $\mathcal{H}_{h_\infty}(\vec{p}) = \begin{pmatrix}\hat{\pi} \\ \hat{\vec{\nu}}^t\end{pmatrix} \vec{p} + \begin{pmatrix}\hat{\vec{\mu}} \\ \hat{\rho}\end{pmatrix}$,

where π, $\hat{\pi}$, $\vec{\mu}$, $\hat{\vec{\mu}}$, $\hat{\vec{\mu}}$, ρ, and $\hat{\rho}$ are the components of Π and $\hat{\Pi}$ Equation 2.2.44.

Proof.

$$\begin{aligned}
\mathcal{D}_{h_\infty}(p) &= \mathcal{D}_{(\vec{0}^t,1)^t}\left(\Pi_{h_\infty \to (\vec{0}^t,1)^t} p\right) = \mathcal{D}_{(\vec{0}^t,1)^t}\left(\begin{pmatrix}\pi & \vec{\nu} \\ \vec{\mu}^t & \rho\end{pmatrix}\begin{pmatrix}\vec{p} \\ \tilde{p}\end{pmatrix}\right) = \\
&= \mathcal{D}_{(\vec{0}^t,1)^t}\left(\begin{pmatrix}\pi\vec{p} + \mu\tilde{x} \\ \vec{\nu}^t\vec{p} + \rho\tilde{p}\end{pmatrix}\right) = \pi\vec{p} + \mu\tilde{p} = \\
&= \begin{pmatrix}\pi & \vec{\mu}\end{pmatrix} p, \qquad (2.2.49)\\
\mathcal{H}_{h_\infty}(\vec{p}) &= \hat{\Pi}^{-1}_{h_\infty \to (\vec{0}^t,1)^t}\mathcal{H}_{(\vec{0}^t,1)^t}(\vec{p}) = \\
&= \hat{\Pi}^{-1}_{h_\infty \to (\vec{0}^t,1)^t}\left(\begin{pmatrix}\mathbb{1}_{d\times d} \\ \vec{0}^t\end{pmatrix}\vec{p} + \begin{pmatrix}\vec{0} \\ 1\end{pmatrix}\right) = \\
&= \begin{pmatrix}\hat{\pi} & \hat{\vec{\mu}} \\ \hat{\vec{\nu}}^t & \hat{\rho}\end{pmatrix}\left(\begin{pmatrix}\mathbb{1}_{d\times d} \\ \vec{0}^t\end{pmatrix}\vec{p} + \begin{pmatrix}\vec{0} \\ 1\end{pmatrix}\right) = \begin{pmatrix}\hat{\pi} \\ \hat{\vec{\nu}}^t\end{pmatrix}\vec{p} + \begin{pmatrix}\hat{\vec{\mu}} \\ \hat{\rho}\end{pmatrix}. \qquad (2.2.50)
\end{aligned}$$

□

One can see from the inverse condition of the $\Pi_{h_\infty \to (\vec{0}^t,1)^t}$ (Equation 2.2.45) that all homogenized points do not lie on the hyperplane at infinity, because the first summand on the right side of Equation 2.2.50 vanishes and the second is always $1 \neq 0$ when executing the dot product with the point in order to test the incidence:

$$\begin{aligned}
\forall\vec{p} : \mathcal{H}_{h_\infty}(\vec{p})^t h_\infty &= \left(\vec{p}^t\begin{pmatrix}\hat{\pi} & \hat{\vec{\nu}}^t\end{pmatrix} + \begin{pmatrix}\hat{\vec{\mu}} & \hat{\rho}\end{pmatrix}\right)\begin{pmatrix}\vec{h}_\infty \\ \tilde{h}_\infty\end{pmatrix} = \\
&= \vec{p}^t(\pi\hat{\vec{\mu}} + \hat{\vec{\mu}}\rho) + (\vec{\nu}^t\hat{\vec{\mu}} + \rho\hat{\rho}) \overset{\text{(Eq. 2.2.47)}}{=} \vec{p}^t\vec{0} + 1 = 1 \neq 0.
\end{aligned}$$

Example 2.68. *Homogenization With Respect to a $h_\infty = (\vec{h}^t_\infty, 1)^t$*

- *In order to retrieve the result from Example 2.49 for homogenized points that are elements of the affine space with respect to hyperplanes of the form $h_\infty = (\vec{h}^t_\infty, 1)^t$ one has to choose the remaining affine transformation that only recoordinatizes the points within the affine space, i.e., $\pi = \mathbb{1}_{d\times d}$ and $\mu = \vec{0}$. $\vec{\nu}^t$ and ρ are* *determined by* h_∞ *to have the values* $\vec{h}_\infty$ *and* 1 *respectively. Thus* $\Pi_{h_\infty \to (\vec{0}^t,1)^t}$ *and its inverse are totally determined and can be plugged into the formulas for the dehomogenization and homogenization (2.67). The inverse has the components* $\hat{\pi} = \mathbb{1}_{d\times d}$, $\hat{\vec{\mu}} = \vec{0}$, $\hat{\vec{\nu}} = -\vec{h}_\infty$, *and* $\hat{\rho} = 1$. *They follow from evaluating the two directions of the inverse condition for* $\Pi_{h_\infty \to (\vec{0}^t,1)^t}$*:*

$$\mathcal{H}_{(\vec{h}^t_\infty,1)^t}(\vec{p}) = \begin{pmatrix} \mathbb{1}_{d\times d} \\ -\vec{h}^t_\infty \end{pmatrix} \vec{p} + \begin{pmatrix} \vec{0} \\ 1 \end{pmatrix} = \begin{pmatrix} \vec{p} \\ 1 - \vec{h}^t_\infty \vec{p} \end{pmatrix} \quad \textit{and} \tag{2.2.51}$$

$$\mathcal{D}_{(\vec{h}^t_\infty,1)^t}(p) = \begin{pmatrix} \mathbb{1}_{d\times d} & \vec{0} \end{pmatrix} \begin{pmatrix} \vec{p} \\ 1 - \vec{h}^t_\infty \vec{p} \end{pmatrix} = \vec{p}. \tag{2.2.52}$$

These explicit tools can now be used to identify the *affine space affinities* (Definition 2.65) with certain *projective space projectivities.*

2.2.3 Affine Transformations Embedded in Projective Spaces

Shifting a vector by a constant vector $\vec{t}$ within the affine space is called a *translation.* This affine transformation can easily be expressed as a projective transformation with respect to the hyperplane $(\vec{0}^t, 1)^t$:

$$T_{(\vec{0}^t,1)^t}(\vec{t}) \begin{pmatrix} \vec{p} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbb{1}_{d\times d} & \vec{t} \\ \vec{0}^t & 1 \end{pmatrix} \begin{pmatrix} \vec{p} \\ 1 \end{pmatrix} = \begin{pmatrix} \vec{p} + \vec{t} \\ 1 \end{pmatrix}. \tag{2.2.53}$$

The general *affinity* in this easy case is also directly given by:

$$A_{(\vec{0}^t,1)^t}(\mathbf{A}, \vec{t}) \begin{pmatrix} \vec{p} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \vec{t} \\ \vec{0}^t & 1 \end{pmatrix} \begin{pmatrix} \vec{p} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\vec{p} + \vec{t} \\ 1 \end{pmatrix}, \tag{2.2.54}$$

where $\mathbf{A}$ can be, e.g., a dilation, rotation, or in general any linear transformation of $\mathbb{F}^d$ with full rank d. One can immediately see that the dehomogenization of a point that got mapped by the affinity of the projective space (Equation 2.2.54) with respect to the easy hyperplane at infinity yields the same result as the affinity defined in Definition 2.65.

The same should hold true for the general representation of an affinity with respect to an arbitrary hyperplane h_∞ at infinity; this conditions renders the idea of *embedding of an affinity as a projectivity* precise:

$$\mathcal{D}_{h_\infty}(A_{h_\infty}(\mathbf{A}, \vec{t})(\mathcal{H}_{h\infty}(\vec{p}))) \stackrel{!}{=} \alpha_{\mathbf{A},\vec{t}}(\vec{p}) = \mathbf{A}\vec{p} + \vec{t}, \tag{2.2.55}$$

$$\mathcal{H}_{h\infty}(\alpha_{\overline{A},\vec{t}}(\mathcal{D}_{h_\infty}(p))) \stackrel{!}{=} A_{h_\infty}(\mathbf{A}, \vec{t})(p), \text{ and} \tag{2.2.56}$$

$$A^{-t}_{h_\infty}(\mathbf{A}, \vec{t})h_\infty = h_\infty \quad \Leftrightarrow \quad A^t_{h_\infty}(\mathbf{A}, \vec{t})h_\infty = h_\infty. \tag{2.2.57}$$

The last condition expresses the fact that the affinity is suppossed to act on the affine space alone and leaves the hyperplane invariant, i.e., that it really is an automorphism of the embedded affine space. But this condition has not to be forced. It can be derived from the resulting form of the embedded affinities. An *explicitely parameterized form* of the affinities can be derived like:

$$A_{h_\infty}(\mathbf{A},\vec{t}) = \Pi_{h_\infty\to(\vec{0}^t,1)^t} A_{(\vec{0}^t,1)^t}(\mathbf{A},\vec{t}) \hat{\Pi}^{-1}_{h_\infty\to(\vec{0}^t,1)^t} = \tag{2.2.58}$$

$$= \begin{pmatrix} \hat{\pi} & \vec{\hat{\mu}} \\ \vec{\hat{\nu}}^t & \hat{\rho} \end{pmatrix} \begin{pmatrix} a & \vec{t} \\ \vec{0}^t & 1 \end{pmatrix} \begin{pmatrix} \pi & \vec{\mu} \\ \vec{h}^t_\infty & \tilde{h}_\infty \end{pmatrix} =$$

$$= \begin{pmatrix} \hat{\pi} & \vec{\hat{\mu}} \\ \vec{\hat{\nu}}^t & \hat{\rho} \end{pmatrix} \left(\mathbb{1} + \begin{pmatrix} (\mathbf{A} - \mathbb{1}_{d\times d}) & \vec{0} \\ \vec{0}^t & 0 \end{pmatrix} + \begin{pmatrix} 0_{d\times d} & \vec{t} \\ \vec{0}^t & 0 \end{pmatrix} \right) \begin{pmatrix} \pi & \vec{\mu} \\ \vec{h}^t_\infty & \tilde{h}_\infty \end{pmatrix} =$$

$$= \mathbb{1} + \begin{pmatrix} \hat{\pi} \\ \vec{\hat{\nu}}^t \end{pmatrix} \left((a - \mathbb{1}_{d\times d}) \begin{pmatrix} \pi & \mu \end{pmatrix} + \vec{t} \begin{pmatrix} \vec{h}_\infty & \tilde{h}_\infty \end{pmatrix} \right). \tag{2.2.59}$$

This calculation proofs the following theorem about how to embed an affinity α into a projective space:

Theorem 2.69. *Affinities $\alpha_{\mathbf{A},\vec{t}}$ Embedded in Projective Space as $A_{h_\infty}(\mathbf{A},\vec{t})$*

For an affinity $\alpha_{\mathbf{A},\vec{t}} : \mathbb{F}_p^d \to \mathbb{F}_p^d$ there is the following embedding into the projective space according to the homogenization with respect to the hyperplane at infinity $h_\infty \in H$:

$$A_{h_\infty}(\mathbf{A},\vec{t}) = \mathbb{1} + \begin{pmatrix} \hat{\pi} \\ \vec{\hat{\nu}}^t \end{pmatrix} \left((\mathbf{A} - \mathbb{1}_{d\times d}) \begin{pmatrix} \pi & \mu \end{pmatrix} + \vec{t} \begin{pmatrix} \overrightarrow{h_\infty} & \tilde{h}_\infty \end{pmatrix} \right). \tag{2.2.60}$$

It follows directly from the inverse condition of the $\Pi_{h_\infty\to(\vec{0}^t,1)^t}$ (Equation 2.2.47) that the projective space affinities leave the hyperplane at infinity invariant:

$$A^t_{h_\infty}(\mathbf{A},\vec{t})h_\infty = \left(\mathbb{1} + \left(\begin{pmatrix} \pi^t \\ \vec{\mu}^t \end{pmatrix} (\mathbf{A}^t - \mathbb{1}) + \begin{pmatrix} \vec{h}^t_\infty \\ \tilde{h}_\infty \end{pmatrix} \vec{t}^t \right) \begin{pmatrix} \hat{\pi}^t & \vec{\nu} \end{pmatrix} \right) \begin{pmatrix} \vec{h}^t_\infty \\ \tilde{h}_\infty \end{pmatrix} = h_\infty, \tag{2.2.61}$$

because $\begin{pmatrix} \hat{\pi}^t & \vec{\nu} \end{pmatrix} \begin{pmatrix} \vec{h}^t_\infty \\ \tilde{h}_\infty \end{pmatrix} = \vec{0}^t$ due to $\hat{\Pi}_{h_\infty\to(\vec{0}^t,1)^t} \Pi^{-1}_{h_\infty\to(\vec{0}^t,1)^t} = \mathbb{1}$. The commutative diagram in Figure 2.8 summarizes the interplay of (de-)homogenizations and affinities.

Mapping Points and Lines Simultaneously in Two Dimensions

Later the considered biquadrics will have a so-called center point p_c and it is relevant, how to generate biquadric fields, i.e., a collection of biquadrics such that there are as many biquadrics as points in the projective geometry and such that all points are a center point of a biquadric. In order to do so I tried to find an explicitly

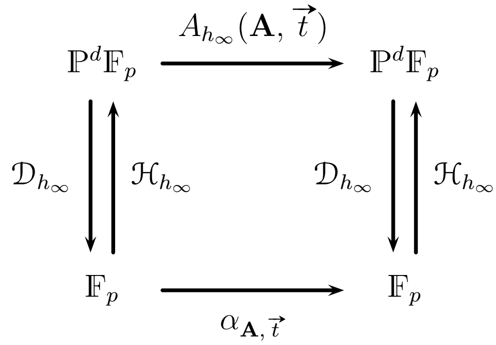


Figure 2.8: A commutative diagram that shows that it does not matter whether one executes the affinity in the affine space $\alpha_{\mathbf{A},\vec{t}}$ first and then homogenizes ($\mathcal{H}_{h_\infty}$) or whether one first homogenizes ($\mathcal{H}_{h_\infty}$) a point and then acts on it with the projective transformation $A^t_{h_\infty}(\mathbf{A}, \vec{t})$ resembling the affinity in the projective space.

parameterized form of the most general transformations in two dimension that map this center point p_c to a new center point p'_c while mapping the corresponding polar $\mathrm{pol}_M(p_c)$ simultaneously to another line, before I found an explicit form of all biquadrics. Hence the question arose how to map a point p onto another point p' while simultaneously map a line ℓ onto another line ℓ'.

It is easy to find the solution for the point $p = (0, 0, 1)^t$ and the line $\ell = (0, 0, 1)^t$ and subsequently generalize it to arbitrary points and lines.

A projective transformation is given by a regular matrix Π. In two dimensions — that is for projective planes — the matrix Π can be parametrized by

$$\Pi = \begin{pmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{pmatrix}, \tag{2.2.62}$$

with a non-vanishing determinant

$$\begin{aligned} \det(M) = &+ \pi_{11}\pi_{22}\pi_{33} + \pi_{12}\pi_{23}\pi_{31} + \pi_{13}\pi_{21}\pi_{32} - \\ &- \pi_{31}\pi_{22}\pi_{13} - \pi_{32}\pi_{23}\pi_{11} - \pi_{33}\pi_{21}\pi_{12} \neq 0. \end{aligned} \tag{2.2.63}$$

To impose that $p' = \Pi p$ with $r' = (p'_1, p'_2, p'_3)^t$ yields

$$\Pi p = \begin{pmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \pi_{13} \\ \pi_{23} \\ \pi_{33} \end{pmatrix} \stackrel{!}{=} p' = \begin{pmatrix} p'_1 \\ p'_2 \\ p'_3 \end{pmatrix}. \tag{2.2.64}$$

Hence the components of the last column of π are determined and one can structure Π like this:

$$\Pi = \begin{pmatrix} | & | & | \\ \pi_1 & \pi_2 & p' \\ | & | & | \end{pmatrix}, \tag{2.2.65}$$

with π_1, $\pi_2 \in \mathbb{F}^d$.

The next condition to impose is the transformation of the line: $\ell' = (\Pi^{-1})^t \ell$. Thus one is interested in the explicite form of the transposed inverse of Π according to Theorem 2.63:

The matrix $\Pi^* = (\Pi^{-1})^t = (\Pi^t)^{-1}$ shall be called *dual matrix* of Π and the following considerations reveal the structure of Π^* and why the attribute "dual" is justified. A representative of Π^* has the form[31]:

$$\Pi^* = (\Pi^{-1})^t = \begin{pmatrix} | & | & | \\ \pi_2 \times p' & p' \times \pi_1 & \pi_1 \times \pi_2 \\ | & | & | \end{pmatrix}. \tag{2.2.66}$$

As for the point it is easy to reduce everything to $\ell = (0, 0, 1)^t$. Then $\ell' = (\Pi^{-1})^t \ell$ yields that the last column of Π^* has to be the new line ℓ':

$$\Pi^* \ell = \begin{pmatrix} | & | & | \\ \pi_2 \times p' & p' \times \pi_1 & \pi_1 \times \pi_2 \\ | & | & | \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \pi_1 \times \pi_2 \overset{!}{=} \ell'. \tag{2.2.67}$$

Thus an interpretation of π_1 and π_2 as two points on the new line p'_1 and p'_2 is justified. This follows from the calculation method of the joint line by the cross-product according to Definition 2.44. Furthermore $\pi_2 \times p'$ and $p' \times \pi_1$ can now be interpreted as two lines through the new point and the two points on the new line ℓ'_1 and ℓ'_2. Figure 2.9 shows this interpretation and the symbols that had been introduced.

Theorem 2.70. *Simultaneous Mapping of a Point p and a Line ℓ*

The most general explicitely parameterized form of the transformation Π that transforms the "standard" point $p = (0, 0, 1)^t$ into an arbitrary point p' and at the same time the "standard" line $\ell = (0, 0, 1)^t$ into an arbitrary new line ℓ' looks as follows for two arbitrary but distinct points on this new line (p'_1 and p'_2):

$$\Pi = \begin{pmatrix} | & | & | \\ p'_1 & p'_2 & p' \\ | & | & | \end{pmatrix} \quad \textit{with} \quad (p'_1 \times p'_2 = \ell') \quad \textit{and} \tag{2.2.68}$$

$$\Pi^* = \begin{pmatrix} | & | & | \\ \ell'_1 & \ell'_2 & \ell' \\ | & | & | \end{pmatrix} \quad \textit{with} \quad (\ell'_1 \times \ell'_2 = p'), \tag{2.2.69}$$

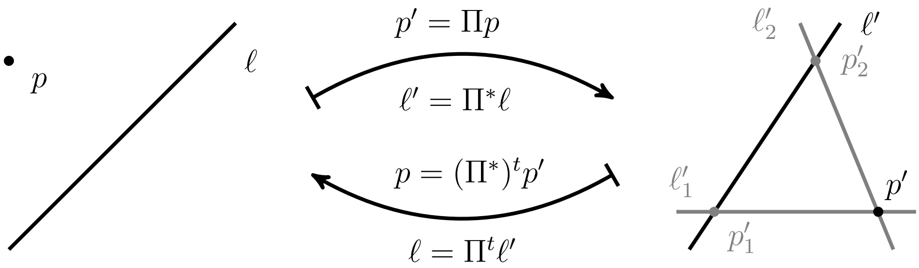


Figure 2.9: In order to transform the standard point $p = (0,0,1)^t$ into an arbitrary new point p' and simultaneously the standard line $\ell = (0,0,1)^t$ into an arbitrary new line ℓ' one has to pick two arbitrary points p'_1 and p'_2 on the new line (depicted in gray) yielding two lines being incident with those points and the new point (ℓ'_1 and ℓ'_2). These lines and points constitute the columns and rows of the matrix Π and its dual matrix Π^* that serve as the transformations in this case.

The duality is obvious now: the three columns of the point transformation are interpretable as three points of a triangle whose dual figure consists of lines that are the columns of the line transformation. This is only valid for the standard line and standard point. But it enables one immediately to switch between arbitrary points (for example p' and p'') and lines (for example ℓ' and ℓ''). One has to transform back to the standard point ($p = (\Pi^*_{s\to\prime})^t p'$) and standard line ($\ell' = (\Pi_{s\to\prime})^t \ell'$) and then transform to the new point ($p'' = (\Pi_{s\to\prime\prime})p = (\Pi_{s\to\prime\prime})(\Pi^*_{s\to\prime})^t p'$) and new line ($\ell'' = (\Pi_{s\to\prime})^t \ell' = (\Pi^*_{s\to\prime\prime})(\Pi_{s\to\prime})^t \ell'$). This reveals an explicetely parameterized and easily interpretable parameterization of the general transformations $\Pi_{\prime\to\prime\prime}$ for transforming an arbitrary point p' into an arbitrary new point p'' and at the same time transform an arbitrary line ℓ' into an arbitrary new line ℓ''. One has to pick two arbitrary points (p'_1 and p'_2) on the old line and two on the new line (p''_1 and p''_2) and calculate the joints of those points and the points (ℓ'_1, ℓ'_2 and ℓ''_1, ℓ''_2) and use them like this as the rows and columns of the transformation matrices:

$$\begin{aligned} p''_c &= \Pi_{\prime\to\prime\prime}\ p' = \Pi_{s\to\prime\prime}\ (\Pi^*_{s\to\prime})^t p'_c \\ \text{with}\quad \Pi_{\prime\to\prime\prime} &= \begin{pmatrix} | & | & | \\ p''_1 & p''_2 & p''_c \\ | & | & | \end{pmatrix} \cdot \begin{pmatrix} - & \ell'_1 & - \\ - & \ell'_2 & - \\ - & \ell' & - \end{pmatrix}, \end{aligned} \tag{2.2.70}$$

$$\begin{aligned} \ell'' &= \Pi^*_{\prime\to\prime\prime}\ \ell' = \Pi^*_{s\to\prime\prime}\ (\Pi_{s\to\prime})^t \ell' \\ \text{with}\quad \Pi^*_{\prime\to\prime\prime} &= \begin{pmatrix} | & | & | \\ \ell''_1 & \ell''_2 & \ell'' \\ | & | & | \end{pmatrix} \cdot \begin{pmatrix} - & p'_1 & - \\ - & p'_2 & - \\ - & p'_c & - \end{pmatrix}, \end{aligned} \tag{2.2.71}$$

[31] Due to the behavior of the cross-product and homogenity.

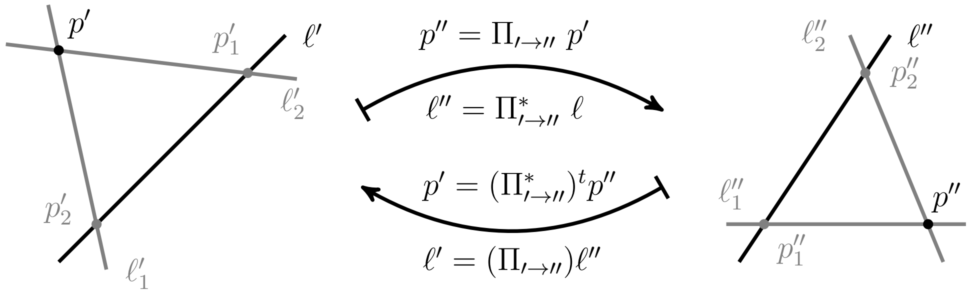


Figure 2.10: To transform an arbitrary point p'_c into an arbitrary new point p'' and an arbitrary line ℓ' into an arbitrary new line ℓ'' one has to pick two arbitrary points p'_1 and p'_2 on the old line and two points p''_1 and p''_2 on the new one (all depicted in gray) yielding four lines being incident with those points and the old and new points each (ℓ'_1, ℓ'_2, ℓ''_1 and ℓ''_2). These lines and points constitute the columns and rows of the matrix $\Pi_{\prime\to\prime\prime}$ and and its dual matrix $\Pi^*_{\prime\to\prime\prime}$ that serve as the transformations in this general case. The inverse transformations are given by the transposed dual matrix each.

while the following relations hold true:

$$\begin{aligned} p'_1 \times p'_2 &= \ell', \\ p''_1 \times p''_2 &= \ell'', \\ l'_1 \times l'_2 &= p', \text{ and} \\ l''_1 \times l''_2 &= p''. \end{aligned} \tag{2.2.72}$$

Figure 2.10 illustrates this general behavior. The “s” in the subscript of the transformations stands for “standard”.

There an important point has to be mentioned: in order to yield all transformations the two first columns of the Π have to be scaled independently by two factors $\in \mathbb{F}\backslash\{0\}$, because homogenity only affects the whole matrix and alone-standing points and hyperplanes and not columns or rows of matrices only because they could be interpreted as points or hyperplanes. This remark is only important if the points and lines that consititue the rows and columns of the transformation matrices are choosen from a set of normalized points and lines.

The apparent difference of the degrees of freedom for the easy case and the general case (once two points to choose on the new line and once four points; two on the old and two on the new) is not real: in the easy case one could place a projective transformation in front that maps from the standard point to the standard point and at the same time from the standard line to the standard line (and here one could choose two points on the standard line as well).

All of this can be generalized to higher dimension using the generalized cross-product and respectively more points on the new hyperplanes.

One can interpret the first d columns of $M_{s\to\prime}$ as d different points on the new hyperplane h' (call them $p'_1, \ldots, p'_d$) and the last column is the new point p'. The cross products in the representative of $M^*_{s\to\prime}$ are then the joint lines of the new point p' and the points $p'_1, \ldots, p'_d$ except one each (call them $h'_1, \ldots, h'_d$) and the last column is the new line h':

$$M_{s\to\prime} = \begin{pmatrix} | & & | & | \\ p'_1 & \ldots & p'_d & p' \\ | & & | & | \end{pmatrix}, \tag{2.2.73}$$

$$M^*_{s\to\prime} = \begin{pmatrix} | & & | & | \\ h'_1 & \ldots & h'_d & h' \\ | & & | & | \end{pmatrix}. \tag{2.2.74}$$

The generalization for transforming an arbitrary point p' into an arbitary point p'' simultaneously to transforming and arbitrary hyperplane at infinity h'_∞ into an arbitrary hyperplane h''_∞ in d dimensions is completely analogously to the same kind of generalization in the two dimensional case (Equation 2.2.71).

2.3 Biquadrics

In the general theory of relativity the dynamical object is the metric tensor $g^{\mu\nu}$ that is a symmetric $\binom{0}{2}$-tensor in each tangent space. Roughly speaking it allows to measure length "point-dependent". For all space- and light-like directions there is a unit length defined by the intersection of a line through that point into that direction with the quadric given by the metric tensor component matrix in each tangent space. If one tries to use a quadric in the same manner to define length in finite projective geometries over Galois fields, a problem arises, because there is no longer an intersection of each line through the point of interest with the quadric for that point. But it is possible to equip a point not only with one quadric, but with two, such that, it is guaranteed, to have an intersection with the quadrics in each "backward" and "forward direction": biquadrics.

Need for Biquadrics

There are ${(1-q^d)}/{(1-q)}$ points on a quadric and there are also ${(1-q^d)}/{(1-q)}$ lines incident with a point for the dimension d in a finite projective geometry over a Galois field $\mathbb{F}_q$ (see p. 147 et seq. in [Beu98]). Thus there are $2 \cdot {(1-q^d)}/{(1-q)}$ directions (two per line). Hence there are ${(1-q^d)}/{(1-q)}$ points missing on one quadric to guarantee an intersection in all directions and the idea is to use two quadrics per point to encode a unit length (at least locally) in all directions: a pair of two quadrics that suffices to yield an intersection in all direction will be called *biquadric*. The thus selected point will be called *center point* p_c. This is illustrated in Figure 2.11.

Definition 2.71. *Biquadric $\mathcal{B}_{p_c}$ (see. p. 25 et seq. in [Ale12])*

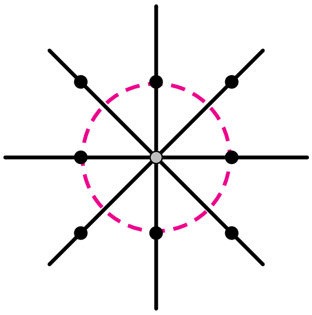
For a point $p_c \in P$ a biquadric is pair of two quadrics $\mathcal{B}_{p_c} = (Q_{M_1}, Q_{M_2})$, s.t., there are two intersection on each line $\ell \in L, p_c \in \ell$[32]. Formally:

- $\forall \ell \in L : \ell^t p_c = 0 \Rightarrow |\ell \cap Q_{M_1}| + |\ell \cap Q_{M_2}| = 2$ *and*
- $p \notin Q_{M_1} \cap Q_{M_1}$.

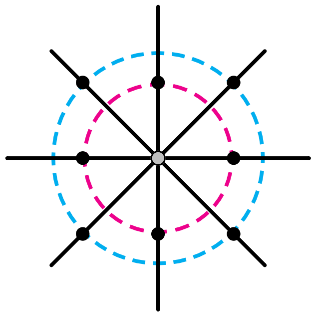
There are two possibilities for the polars of biquadrics: either both quadrics of the quadric pair have the same polar or there are different polars. Numerical simulations have proven that both cases exist (see Chapter 3.5.3).

In order to formulate biquadrics in an explicitely parameterized form the notion of the "quadratic coset" equivalence relation is handy. This itself depends on the notion of a *coset*:

[32] The lines L are all one-dimensional linear subspace $\mathscr{U}_1 = \mathscr{L} =: L$.



(a) For this affine plane of the projective plane $\mathbb{P}^2\mathbb{F}_3$ that consists of $3^2 = 9$ points an elliptic quadric (red) is shown. Obviously does not suffice to encode a point in each direction. The dashed line is purely symbolic; only the four connected dots constitute the quadric.



(b) The second quadric (cyan) suffices in supplementing the first quadric (magenta), s.t. there is a point in all directions.

Figure 2.11: On the left there is single quadric that lacks points in some directions. Therefore a second quadric is neccessary that is shown on the right. This is the intuition that is rendered precise in Definition 2.71.

Definition 2.72. *Left Cosets gH of a Subgroup H of a Group G*

Let $(G, \circ)$ be a group and H a subgroup of G[33] *and $\sim_H$ the equivalence relation*

$$g_1 \sim g_2 \quad :\Leftrightarrow \quad g_1^{-1} \circ g_2 \in H \tag{2.3.75}$$

then the coset gH is to be defined the corresponding equivalence class

$$gH := [g]_{\sim_H}. \tag{2.3.76}$$

Definition 2.73. *Quadratic $\mathcal{Q}(\mathbb{F})$ and Non-Quadratic Coset $\bar{\mathcal{Q}}(\mathbb{F})$*

For any field $(\mathbb{F}, +, \cdot)$ there is the multiplicative group $(\mathbb{F}\backslash\{0\}, \cdot)$. $H = \{f^2 | f \in \mathbb{F}\backslash\{0\}\}$ is the subgroup that arises, if all elements of $\mathbb{F}\backslash\{0\}$ are squared. Then the two cosets are

- *Quadratic Coset* $\mathcal{Q}(\mathbb{F}) := [1]_H = 1H = H$
  (All elements that possess a square root[34]*.) and*
- *Non-Quadratic Coset* $\bar{\mathcal{Q}}(\mathbb{F}) := (\mathbb{F}\backslash\{0\})\backslash\mathcal{Q}(\mathbb{F})$
  (All elements that do not possess a square root.).

[33] I.e., $H \subset G$ and $(H, \circ_{|H})$ is a group.

[34] Where "square root" of a number $\lambda \in \mathbb{F}_p\backslash\{0\}$ simply means that a number $\xi \in \mathbb{F}_p\backslash\{0\}$ exists that squared yields the number. Then there is a second number with that property $(-\xi)$.

Explicitely Parameterized Form of Biquadrics

For the case that the two quadrics of the biquadric share one polar, the "standard polar" $\mathrm{pol}_{M_{1\backslash 2}} = (\vec{0}^{\,t}, 1)^t$ and that the center point is the "standard center point" $p_c = (\vec{0}^{\,t}, 1)^t$, the form of biquadrics is known:

Theorem 2.74. *Biquadrics $\mathcal{B}_{(\vec{0}^t,1)^t,(\vec{0}^t,1)^t}$ (see p. 25 et seq. in [Ale12])*

For the <u>center point</u> $p_c = (\vec{0}^{\,t}, 1)^t$ and the <u>unique polar</u> $pol_{M_{1\backslash 2}} = (\vec{0}^{\,t}, 1)^t$ the matrices $\mathcal{M}_{(\vec{0}^t,1)^t,(\vec{0}^t,1)^t} = (M_1, M_2)$ are the corresponding matrices of the biquadrics $\mathcal{B}_{(\vec{0}^t,1)^t;(\vec{0}^t,1)^t}$, where $A \in \mathbb{F}_p^{d\times d}$ (with $\det(A) \neq 0$) and for all $\bar{q} \in \bar{\mathcal{Q}}(\mathbb{F}_p)$ and $q \in \mathcal{Q}(\mathbb{F}_p)$:

$$\mathcal{M}_{(\vec{0}^t,1)^t;(\vec{0}^t,1)^t}(A, q, \bar{q}) = \left(\begin{pmatrix} qA & \vec{0} \\ \vec{0}^t & 1 \end{pmatrix}, \begin{pmatrix} \bar{q}A & \vec{0} \\ \vec{0}^t & 1 \end{pmatrix} \right). \qquad (2.3.77)$$

The last column and row of both matrices have to be the common polar $(\vec{0}^{\,t}, 1)^t$ due to the two conditions $M_1 p_c = M_1(\vec{0}^{\,t}, 1)^t = \mathrm{pol}_{M_{1\backslash 2}}(p_c) = (\vec{0}^{\,t}, 1)^t$ and $M_2 p_c = M_2(\vec{0}^{\,t}, 1)^t = \mathrm{pol}_{M_{1\backslash 2}}(p_c) = (\vec{0}^{\,t}, 1)^t$ and the symmetry of the matrices. $\det(M_1) = q^d \cdot \det(A) = 0$ and $\det(M_2) = \bar{q}^d \cdot \det(A) = 0$ are thus guaranteed if an only if $\det(A) = 0$. Hence the pair that is to be constructed has only to obey that $\det(A) \neq 0$. For each $q \in \mathcal{Q}(\mathbb{F}_p)$ one can choose all of the $\bar{q} \in \bar{\mathcal{Q}}(\mathbb{F}_p)$ resulting in a biquadric. Thus for a given matrix A there are two classes of quadrics. Those where A gets scaled by a number that possess a square root and those where A gets scaled with a number that does not possess a square root. All members of the first class combined with an arbitrary member of the second class yield a biquadric[35] (see p. 31 in [Ale12]).

The proof of the two dimensional case can be generalized to arbitrary dimensions. But because this is included in the general proof for biquadrics with respect to arbitrary center points and polars, the proof of this special case is ommited here. But the idea is to show that M_2 leads to a quadric that has the same tangents as Q_{M_1}, but exactely all secants of Q_1 are turned into passants of Q_2 and vice versa (see Theorem 2.77).

Example 2.75. *Biquadric for Center Point $p_c = (0, 0, 1)^t$ and Polar $\mathrm{pol}_{M_{1\backslash 2}}(p_c) = (0, 0, 1)^t$*

For the finite projective plane $\mathbb{P}^2\mathbb{F}_3$ the following set of matrices underlies a biquadric $\mathcal{B}_{(\vec{0}^t,1)^t;(\vec{0}^t,1)^t}$:

$$\mathcal{M}_{(\vec{0}^t,1)^t;(\vec{0}^t,1)^t}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 1, 2 \right) = \left(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right). \qquad (2.3.78)$$

[35] This is why the factor q is displayed explicetely; because in principle it could have been absorbed into the matrix A.

The biqudric intersects the polar twice and thus it is called hyperbolic. *A practical condition in order to check whether a biquadric is hyperbolic or elliptic will be given in Theorem 2.76. The biquadric consists of the points:*

$$\mathcal{B}_{(\vec{0}^t,1)^t;(\vec{0}^t,1)^t}\left(\begin{pmatrix}0&1\\1&0\end{pmatrix},1,2\right)=\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix};\right.$$
$$\left.\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}. \tag{2.3.79}$$

It is illustrated in Figure 2.12.

Besides the practicality and elegance of explicite analytic forms on the purely mathematical side, it is also benefitial to have access to an explicitely parameterized form for simulations sake, because it is of "the order of plugging in numbers".

Therefore the interesting question is: how does the submatrix B look in the following pair of quadrics, s.t., the pair is a biquadric:

$$\mathcal{M}_{(\vec{0}^t,1)^t;h_\infty}=\left(\begin{pmatrix}qA&\vec{h}_\infty\\\vec{h}^t_\infty&1\end{pmatrix},\begin{pmatrix}B&\vec{h}'_\infty\\(\vec{h}')^t_\infty&1\end{pmatrix}\right), \tag{2.3.80}$$

where $h_\infty=(\vec{h}^t_\infty,1)^t:=\mathrm{pol}_{M_{1\backslash 2}}(p_c)$, because it is interpreted as a hyperplane at infinity and in order to lighten the notation.

The key discovery that leads to the generalization of Theorem 2.74 lies in a condition to determine the relative position of the center point to the biquadric. If it lies outside of both quadrics of the biquadric, the biquadric is called *hyperbolic* and if lies inside it is called *elliptic*. The condition in the standard case (both the center point and the polar have the easy form $(\vec{0}^t,1)^t$) for *hyperbolicity* is, that $-\det(A)\in\mathcal{Q}(\mathbb{F})$, because only then the square root in the solution formula of a second order polynomials does exist (see p. 29 in [Ale12]). In order to generalize this for matrices of the type above, the following "inspirational calculation" turned up for the two dimensional case. It lead to Theorem 2.76:

$$\begin{aligned}\det\left(\begin{pmatrix}A&\vec{\ell}_\infty\\\vec{\ell}^t_\infty&1\end{pmatrix}\right)&=\det\left(\begin{pmatrix}a&b&\ell_1\\b&c&\ell_2\\\ell_1&\ell_2&1\end{pmatrix}\right)=\\&=ac+b\ell_1\ell_2+b\ell_1\ell_2-c\ell_1^2-a\ell_2^2-b^2=\\&=a(c-\ell_2^2)-(b-\ell_1\ell_2)^2+\ell_1^2(\ell_2^2-c)=\\&=(a-\ell_1^2)(c-\ell_2^2)-(b-\ell_1\ell_2)^2=\det\left(A-\vec{\ell}_\infty\vec{\ell}^t_\infty\right)=\\&=\det\left(\begin{pmatrix}A-\vec{\ell}_\infty\vec{\ell}^t_\infty&\vec{0}\\\vec{0}^t&1\end{pmatrix}\right).\end{aligned} \tag{2.3.81}$$

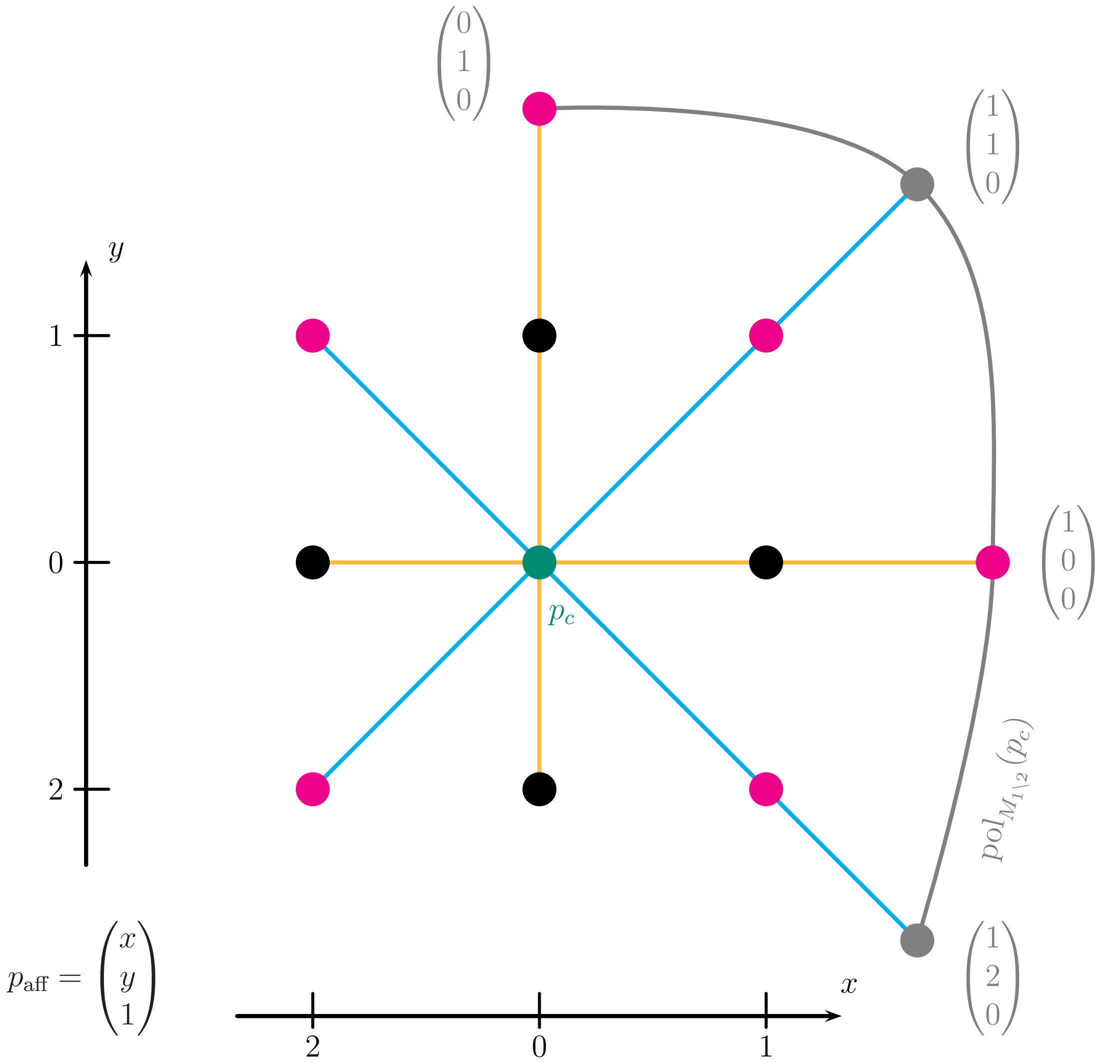


Figure 2.12: A hyperbolic biquadric (from Example 2.75) in the finite projective plane $\mathbb{P}^2\mathbb{F}_3$. There are two tangents (yellow) that intersect the biquadric *at infinity* and two secants (blue) that intersect the biquadric in the affine plane. These are all lines that are incident with the center point p_c (green). The polar $\mathrm{pol}_{M_{1\backslash 2}}(p_c)$ (gray) serves as line at infinity. the two biquadric points on the polar lie in both individual quadrics of the pair and thus are to be counted twice according to Theorem 2.74.

The determinant of the sum of a matrix with another matrix is decomposable as a dyadic product as

$$\begin{pmatrix} \mathbb{1} & \vec{0} \\ \vec{\alpha}^t & 1 \end{pmatrix} \begin{pmatrix} \mathbb{1} + \vec{\beta}\vec{\alpha}^t & \vec{\beta} \\ \vec{0}^t & 1 \end{pmatrix} \begin{pmatrix} \mathbb{1} & \vec{0} \\ -\vec{\alpha}^t & 1 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & \vec{0} \\ \vec{\alpha}^t & 1 \end{pmatrix} \begin{pmatrix} \mathbb{1} & \vec{\beta} \\ -\vec{\alpha}^t & 1 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & \vec{\beta} \\ \vec{0}^t & 1 + \alpha^t \beta \end{pmatrix}$$

for $\vec{\alpha}$ and $\vec{\beta}$ two arbitrary vectors within the same vector space. Evaluating the determinant of both sides yields:

$$\det(\mathbb{1} + \vec{\beta}\vec{\alpha}^t) = 1 + \vec{\alpha}^t \vec{\beta}. \tag{2.3.82}$$

For a matrix A fitting to the dimension of the two vectors this finally can be generalized to

$$\det(A + \vec{\beta}\vec{\alpha}^t) = \det(A) \cdot (\mathbb{1} + A^{-1}\vec{\beta}\vec{\alpha}^t) = \det(A) \cdot (1 + \vec{\alpha}^t A^{-1} \vec{\beta}). \tag{2.3.83}$$

Thus Equation 2.3.81 can be brought into another form:

$$-\det\left(A - \vec{\ell}_\infty \vec{\ell}^{\,t}_\infty\right) = -\det(A)\left(1 - \vec{\ell}^{\,t}_\infty A^{-1} \vec{\ell}_\infty\right). \tag{2.3.84}$$

Hence the following theorem holds true:

Theorem 2.76. *Relative Position of* $p_c = (\vec{0}^t, 1)^t$ *to* $\mathcal{B}_{(\vec{0}^t 1)^t, (\vec{\ell}^{\,t}_\infty)^t}$

For $\mathbb{P}^2\mathbb{F}_p$*, i.e., in the projective plane:*

- $\mathcal{B}_{(\vec{0}^t,1)^t,(\vec{0}^t,1)^t}$ *is hyperbolic* $:\Leftrightarrow -det(A) \in \mathcal{Q}(\mathbb{F}_p)$*; otherwise it is elliptic,*
- $\mathcal{B}_{(\vec{0}^t,1)^t,(\vec{\ell}^{\,t}_\infty,1)^t}$ *is hyperbolic* $:\Leftrightarrow -det\left(A - \vec{\ell}_\infty \vec{\ell}^{\,t}_\infty\right) \in \mathcal{Q}(\mathbb{F}_p)$*, or*
- $p_c = (\vec{0}^t, 1)^t$ *has the same relative position to* $\mathcal{B}_{(\vec{0}^t,1)^t,(\vec{\ell}^{\,t}_\infty 1)^t}$ *as it has relative to* $\mathcal{B}_{(\vec{0}^t,1)^t,(\vec{0}^t,1)^t}$ $:\Leftrightarrow \left(1 - \vec{\ell}^{t}{}_\infty A^{-1} \vec{\ell}_\infty\right) \in \mathcal{Q}(\mathbb{F}_p)$[36].

This result might be true in higher dimensions, but up to now it is only proven for two dimensions; but it was the starting point for a row of suggestions for the general form of biquadrics in arbitrary dimensions that finally lead to:

[36] Note that this bracket contains the term that should vanish in case that $\vec{\ell}_\infty$ lies on the dual biquadric to $\mathcal{B}_{(\vec{0}^t,1)^t,(\vec{0}^t,1)^t}$, that is the one belonging to the inverted matrix; but further interpretations at this oberservation would be interesting.

Theorem 2.77. *All Biquadrics* $\mathcal{B}_{(\vec{0}^t,1)^t;h_\infty}$ *With Respect to All* $h_\infty = (\vec{h}^t_\infty, 1)^t$

For the center point $p_c = (\vec{0}^t, 1)^t$ *the following matrices* $\mathcal{M}_{(\vec{0}^t,1)^t;h_\infty} = (M_1, M_2)$ *are the corresponding matrices of a biquadrics* $\mathcal{B}_{(\vec{0}^t,1)^t}$ *for all* $A \in \mathbb{F}_p^d$*, for all* $(\vec{h}^t_\infty, 1)^t \in \mathbb{P}^d\mathbb{F}_p^*$*, and for all* $\bar{q} \in \bar{\mathcal{Q}}(\mathbb{F}_p)$ *and* $q \in \mathcal{Q}(\mathbb{F})$*, s.t., det*$(M_1) \neq 0$ *and det*$(M_2) \neq 0$*:*

$$\mathcal{M}_{(\vec{0}^t,1)^t;h_\infty}(A, q, \bar{q}) = \left(\begin{pmatrix} q\left(A - \vec{h}_\infty \vec{h}^t_\infty\right) + \vec{h}_\infty \vec{h}^t_\infty & \vec{h}_\infty \\ \vec{h}^t_\infty & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} \bar{q}\left(A - \vec{h}_\infty \vec{h}^t_\infty\right) + \vec{h}_\infty \vec{h}^t_\infty & \vec{h}_\infty \\ \vec{h}^t_\infty & 1 \end{pmatrix} \right). \qquad (2.3.85)$$

The pair can alternatively be given in this form:

$$\mathcal{M}_{(\vec{0}^t,1)^t;h_\infty}(A, q, \bar{q}) = \left(\left(q \begin{pmatrix} \mathbb{1}_{d\times d} \\ \vec{0}^t \end{pmatrix} (A - \vec{h}_\infty \vec{h}^t_\infty) \begin{pmatrix} \mathbb{1}_{d\times d} & \vec{0}^t \end{pmatrix} + h_\infty h^t_\infty \right), \right.$$
$$\left. \left(\bar{q} \begin{pmatrix} \mathbb{1}_{d\times d} \\ \vec{0}^t \end{pmatrix} (A - \vec{h}_\infty \vec{h}^t_\infty) \begin{pmatrix} \mathbb{1}_{d\times d} & \vec{0}^t \end{pmatrix} + h_\infty h^t_\infty \right) \right). \qquad (2.3.86)$$

This form suggests that the biquadric arises from homogenizing an affine quadric, once scaled with a square number and once with a number that does not possess a square number. The last term in both matrices is a projection onto the polar that is interpreted as a hyperplane at infinity[37]. This intuition is based on the form of the matrix representation of the dehomogenization and homogenization process of Theorem 2.67.

The same two classes arise as in the standard case: scaling the first summand of the first matrix in Equation 2.3.86 with an arbitrary scalar factor that *possess* a square root q leads in combination with the second matrix of which the first summand can be scaled with an arbitrary scalar factor that *does not possess* a square root $\bar{q}$ always a biquadric.

[37]But this intuition will have to be made precise in upcoming work.

Proof. In order to proof this, one has to show that tangents of Q_1 are tangents of Q_2 and that secants (two intersections) of Q_1 are passents (no intersection) of Q_2 and vice versa.

Their tangents are the lines through the center point p_c and the intersections of the quadric with the hyperplane at infinity (the polar). These intersections lies on the hyperplane at infinity. For $h_\infty = (\vec{h}^t_\infty, 1)^t$ the points on h_∞ have the form $p_\infty = (\vec{p}^t_\infty, -\vec{h}^t_\infty \vec{p}^t_\infty)$, s.t., $h^t_\infty p_\infty = 0$. Plugging these into the condition for a point lying on the quadric yields for Q_1:

$$\begin{aligned}
0 &\overset{!}{=} p^t_\infty M_1 p_\infty \\
&= \begin{pmatrix} \vec{p}^t_\infty & -\vec{h}^t_\infty \vec{p}_\infty \end{pmatrix} \left(q \begin{pmatrix} \mathbb{1}_{d\times d} \\ \vec{0}^t \end{pmatrix} (A - \vec{h}_\infty \vec{h}^t_\infty) \begin{pmatrix} \mathbb{1}_{d\times d} & \vec{0}^t \end{pmatrix} + h_\infty h^t_\infty \right) \begin{pmatrix} \vec{p}_\infty \\ -\vec{h}^t_\infty \vec{p}_\infty \end{pmatrix} \\
&= \begin{pmatrix} \vec{p}^t_\infty & -\vec{h}^t_\infty \vec{p}_\infty \end{pmatrix} \left(q \begin{pmatrix} \mathbb{1}_{d\times d} \\ \vec{0}^t \end{pmatrix} (A - \vec{h}_\infty \vec{h}^t_\infty) \begin{pmatrix} \mathbb{1}_{d\times d} & \vec{0}^t \end{pmatrix} \right) \begin{pmatrix} \vec{p}_\infty \\ -\vec{h}^t_\infty \vec{p}_\infty \end{pmatrix} + 0 \\
&= q \vec{p}^t_\infty \left(A - \vec{h}_\infty \vec{h}^t_\infty \right) \vec{p}_\infty. \qquad (2.3.87)
\end{aligned}$$

The following calculation shows that the same point that lies on the hyperplane at infinity plugged into the condition to lie on the second quadric Q_2 also yields zero and thus the tangents of Q_1 are also the tangents of Q_2:

$$\begin{aligned}
&p^t_\infty M_2 p_\infty \\
&= \begin{pmatrix} \vec{p}^t_\infty & -\vec{h}^t_\infty \vec{p}_\infty \end{pmatrix} \left(\bar{q} \begin{pmatrix} \mathbb{1}_{d\times d} \\ \vec{0}^t \end{pmatrix} (A - \vec{h}_\infty \vec{h}^t_\infty) \begin{pmatrix} \mathbb{1}_{d\times d} & \vec{0}^t \end{pmatrix} + h_\infty h^t_\infty \right) \begin{pmatrix} \vec{p}_\infty \\ -\vec{h}^t_\infty \vec{p}_\infty \end{pmatrix} \\
&= \begin{pmatrix} \vec{p}^t_\infty & -\vec{h}^t_\infty \vec{p}_\infty \end{pmatrix} \left(\bar{q} \begin{pmatrix} \mathbb{1}_{d\times d} \\ \vec{0}^t \end{pmatrix} (A - \vec{h}_\infty \vec{h}^t_\infty) \begin{pmatrix} \mathbb{1}_{d\times d} & \vec{0}^t \end{pmatrix} \right) \begin{pmatrix} \vec{p}_\infty \\ -\vec{h}^t_\infty \vec{p}_\infty \end{pmatrix} + 0 \\
&= \bar{q} \vec{p}^t_\infty \left(A - \vec{h}_\infty \vec{h}^t_\infty \right) \vec{p}_\infty \overset{\text{(Eq. 2.3.87)}}{=} \bar{q} \cdot 0 = 0 \quad \Rightarrow \quad p_\infty \in Q_2. \qquad (2.3.88)
\end{aligned}$$

It remains to show that passants turn into secants and vice versa. A secant intersects the quadric twice in the affine plane with respect to the polar $h_\infty = (\vec{h}^t_\infty, 1)^t$. Points in the affine plane have the form $p_{\text{aff}} = (\vec{p}^t_{\text{aff}}, 1 - \vec{h}^t_\infty \vec{p}_{\text{aff}})^t$. Plugging these into the condition to lie on the first quadric Q_1 yields:

$$
\begin{aligned}
0 &\stackrel{!}{=} p_{\text{aff}}^t M_1 p_{\text{aff}} \\
&= \left(\vec{p}_{\text{aff}}^{\,t} \quad 1-\vec{h}_{\infty}^{\,t}\vec{p}_{\text{aff}}\right)\left(q\begin{pmatrix}\mathbb{1}_{d\times d}\\ \vec{0}^{\,t}\end{pmatrix}(A-\vec{h}_{\infty}\vec{h}_{\infty}^{\,t})\left(\mathbb{1}_{d\times d} \quad \vec{0}^{\,t}\right)+h_{\infty}h_{\infty}^t\right)\begin{pmatrix}\vec{p}_{\text{aff}}\\ 1-\vec{h}_{\infty}^{\,t}\vec{p}_{\text{aff}}\end{pmatrix} \\
&= q\vec{p}_{\text{aff}}^{\,t}\left(A-\vec{h}_{\infty}\vec{h}^{t}_{\infty}\right)\vec{p}_{\text{aff}} \\
&\quad+\left(\vec{p}_{\text{aff}}^{\,t} \quad 1-\vec{h}_{\infty}^{\,t}\vec{p}_{\text{aff}}\right)\begin{pmatrix}\vec{h}_{\infty}\\ 1\end{pmatrix}\left(\vec{h}_{\infty}^{\,t} \quad 1\right)\begin{pmatrix}\vec{p}_{\text{aff}}\\ 1-\vec{h}_{\infty}^{\,t}\vec{p}_{\text{aff}}\end{pmatrix} \\
&= q\vec{p}_{\text{aff}}^{\,t}\left(A-\vec{h}_{\infty}\vec{h}^{t}_{\infty}\right)\vec{p}_{\text{aff}} \\
&\quad+\left(\vec{h}_{\infty}^{\,t}\vec{p}_{\text{aff}}+1-\vec{h}_{\infty}^{\,t}\vec{p}_{\text{aff}}\right)\left(\vec{h}_{\infty}^{\,t}\vec{p}_{\text{aff}}+1-\vec{h}_{\infty}^{\,t}\vec{p}_{\text{aff}}\right) \\
&= q\vec{p}_{\text{aff}}^{\,t}\left(A-\vec{h}_{\infty}\vec{h}^{t}_{\infty}\right)\vec{p}_{\text{aff}}+1.
\end{aligned}
\tag{2.3.89}
$$

The point $p'_{\text{aff}} = (-\vec{p}_{\text{aff}}^{\,t}, 1+\vec{h}_{\infty}^{\,t}\vec{p}_{\text{aff}})^t$ is the point that arises by mirroring the point $p_{\text{aff}} = (\vec{p}_{\text{aff}}^{\,t}, 1-\vec{h}_{\infty}^{\,t}\vec{p}_{\text{aff}})^t$ at the point $p_c = (\vec{0}^{\,t}, 1)^t$ in the affine space with respect to the hyperplane h_∞. This point obviously lies on the same quadric, because after inserting this point into Equation 2.3.89 the two minus signs pull infront and cancel each other. Thus this yields as a lemma that the quadric points lie symmetrically in the induced affine space with respect to $p_c = (\vec{0}^{\,t}, 1)^t$ for all h_∞ on which p_c does not lie and because no point is distinguished in the projective geometry this symmetry property holds true for all points and all respective polars interpreted as hyperplanes at infinity:

$$
\begin{aligned}
-\vec{p}_{\text{aff}}^{\,t} q\left(A-\vec{h}_{\infty}\vec{h}_{\infty}^{\,t}\right)-\vec{p}_{\text{aff}}+1 = \vec{p}_{\text{aff}}^{\,t} q\left(A-\vec{h}_{\infty}\vec{h}_{\infty}^{\,t}\right)\vec{p}_{\text{aff}}+1 = 0 \\
\Rightarrow \quad (p_{\text{aff}} \in Q_1 \quad \Leftrightarrow p'_{\text{aff}} \in Q_1).
\end{aligned}
\tag{2.3.90}
$$

The same holds true for Q_2, because the only difference in the calculation would be that the q has to replaced by $\bar{q}$ and this has no effect on this result.

Lemma 2.78. *Symmetry of the Center Point in the Induced Affine Plane for Quadrics with a Unique Center Point*

In the affine plane induced by selecting the polar h_∞ of the center point p_c as the hyperplane at infinity it holds true that if $p'_{\text{aff}} = (\vec{p}_{\text{aff}}^{\,t}, 1-\vec{h}_{\infty}^{\,t}\vec{p}_{\text{aff}})^t$ lies on the quadric, then also $p'_{\text{aff}} = (-\vec{p}_{\text{aff}}^{\,t}, 1+\vec{h}_{\infty}^{\,t}\vec{p}_{\text{aff}})^t$ does[38]*, i.e., if the center point is dehomogenized as $\vec{0}$ the affine biquadric points are centrally symmetric with respect to that point*[39].

[38]Mind that $p \neq -p'$. Moreover they would be equal. The minus sign is only applied to the dehomogenized part.

[39]If the center point p_c is dehomogenized somewhat differently the biquadric points lie symmetric with respect to that point, which is clear from the fact that projective transformations mediate between both scenarios.

The secant that is spanned by this intersection point p_{aff} and the center point p_c contains in the affine space all the points $p_{\text{aff},\lambda} = (\lambda \vec{p}^{\,t}_{\text{aff}}, 1 - \lambda \vec{h}^{\,t}_{\infty} \vec{p}_{\text{aff}})^t$ for all $\lambda \in \mathbb{F}_p \backslash \{0\}$. In case secants of Q_1 are tangents of Q_2, none of these points should lie on Q_2. This shall be proven by reductio ad absurdum. Therefore it is assumed that there exists a $\lambda \in \mathbb{F}_p \backslash \{0\}$, s.t., the point $p_{\text{aff},\lambda} = (\lambda \vec{p}^{\,t}_{\text{aff}}, 1 - \lambda \vec{h}^{\,t}_{\infty} \vec{p}_{\text{aff}})^t$ lies on Q_2:

$$
\begin{aligned}
0 &\overset{!}{=} p^t_{\text{aff}} M_2 p_{\text{aff}} \\
&= \begin{pmatrix} \lambda \vec{p}^{\,t}_{\text{aff}} & 1 - \lambda \vec{h}^{\,t}_{\infty} \vec{p}_{\text{aff}} \end{pmatrix} \left(\bar{q} \begin{pmatrix} \mathbb{1}_{d\times d} \\ \vec{0}^{\,t} \end{pmatrix} (A - \vec{h}_{\infty} \vec{h}^{\,t}_{\infty}) \begin{pmatrix} \mathbb{1}_{d\times d} & \vec{0}^{\,t} \end{pmatrix} + h_{\infty} h^t_{\infty} \right) \begin{pmatrix} \lambda \vec{p}_{\text{aff}} \\ 1 - \lambda \vec{h}^{\,t}_{\infty} \vec{p}_{\text{aff}} \end{pmatrix} \\
&= \bar{q} \lambda^2 \vec{p}^{\,t}_{\text{aff}} \left(A - \vec{h}_{\infty} \vec{h}^{\,t}_{\infty} \right) \vec{p}_{\text{aff}} \\
&\quad + \begin{pmatrix} \lambda \vec{p}^{\,t}_{\text{aff}} & 1 - \lambda \vec{h}^{\,t}_{\infty} \vec{p}_{\text{aff}} \end{pmatrix} \begin{pmatrix} \vec{h}_{\infty} \\ 1 \end{pmatrix} \begin{pmatrix} \vec{h}^{\,t}_{\infty} & 1 \end{pmatrix} \begin{pmatrix} \lambda \vec{p}_{\text{aff}} \\ 1 - \lambda \vec{h}^{\,t}_{\infty} \vec{p}_{\text{aff}} \end{pmatrix} \\
&= \bar{q} \lambda^2 \vec{p}^{\,t}_{\text{aff}} \left(A - \vec{h}_{\infty} \vec{h}^{\,t}_{\infty} \right) \vec{p}_{\text{aff}} \\
&\quad + \left(\lambda \vec{h}^{\,t}_{\infty} \vec{p}_{\text{aff}} + 1 - \lambda \vec{h}^{\,t}_{\infty} \vec{p}_{\text{aff}} \right) \left(\lambda \vec{h}^{\,t}_{\infty} \vec{p}_{\text{aff}} + 1 - \lambda \vec{h}^{\,t}_{\infty} \vec{p}_{\text{aff}} \right) \\
&= \bar{q} \lambda^2 \vec{p}^{\,t}_{\text{aff}} \left(A - \vec{h}_{\infty} \vec{h}^{\,t}_{\infty} \right) \vec{p}_{\text{aff}} + 1 \overset{\text{(Eq. 2.3.89)}}{=} \bar{q} \lambda^2 \vec{p}^{\,t}_{\text{aff}} \left(A - \vec{h}_{\infty} \vec{h}^{\,t}_{\infty} \right) \vec{p}_{\text{aff}} + 1 = \\
&= -\lambda^2 \bar{q} + 1
\end{aligned}
\tag{2.3.91}
$$

leading to

$$
\bar{q} = \left(\frac{1}{\lambda} \right)^2 \cdot q = \left(\frac{r}{\lambda} \right)^2, \tag{2.3.92}
$$

where $r \in \mathbb{F}_p \backslash \{0\}$ is a root of q that has to exist because $q \in \mathcal{Q}(\mathbb{F}_p)$. This is a contradiction, because $\bar{q} \in \bar{\mathcal{Q}}(\mathbb{F}_p)$ while the right hand side $({}^r/_\lambda)^2 \in \mathcal{Q}(\mathbb{F}_p)$ and both cosets are disjunct equivalence classes! Hence there is no λ such that the point $p_{\text{aff},\lambda} = (\lambda \vec{p}^{\,t}_{\text{aff}}, 1 - \lambda \vec{h}^{\,t}_{\infty} \vec{p}_{\text{aff}})^t$ lies on Q_2 and thus the line with these points $p_{\text{aff},\lambda}$ is a passent of Q_2. This is true for all secants of Q_1 and hence the remaining lines that were passants of Q_1 are secants of Q_2 (all tangents of Q_1 are tangents of Q_2). □

These are all biquadrics for the center point $p_c = (\vec{0}^{\,t}, 1)^t$. Now it is possible to shift all of these using translations with respect to a hyperplane at infinity $(\vec{h}^{\,t}_{\infty}, 1)^t$ to all points that lie in the corresponding affine spaces. This yields all biquadrics for all center points that do not lie on hyperplanes at infinity of the form $(\vec{h}^{\,t}_{\infty}, 1)^t$.

Example 2.79. *Biquadric for Center Point* $p_c = (0,0,1)^t$ *and Polar* $\text{pol}_{M_{1\backslash 2}}(p_c) = (1,1,1)^t$

As in Example 2.75 for the finite projective plane $\mathbb{P}^2\mathbb{F}_3$ *the following set of matrices underlies another biquadric* $\mathcal{B}_{(0,0,1)^t;(1,1,1)^t}$. *It results from the same input data* A, q, *and* $\bar{q}$, *but leads to an elliptic biquadric this time:*

$$
\mathcal{M}_{(0,0,1)^t;(1,1,1)^t} \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 1, 2 \right) = \left(\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right). \tag{2.3.93}
$$

The biqudric intersects the polar not at all and thus it is called ellitpic. This is also be in accordance to the practical condition from Theorem 2.76:

$$-\det\left(A - \vec{\ell}_\infty \vec{\ell}^{\,t}_\infty\right) = -\det\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\right) = -\det\left(\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right) = 2 \notin \mathcal{Q}(\mathbb{F}_3). \tag{2.3.94}$$

The biquadric consists of the points:

$$\mathcal{B}_{(0,0,1)^t;(1,1,1)^t}\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, 1, 2\right) = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}; \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}\right\}. \tag{2.3.95}$$

This biquadric is illustrated in Figure 2.13.

Finally one can start with biquadrics for all the center points $p_c = (0, \ldots, 0, 1, 0)^t$, $p_c = (0, \ldots, 1, 0, 0)$, …, $p_c = (1, \vec{0})^t$ where the only non-vanishing component is a one that is shifted to all possible positions and the respective hyperplanes at infinity that have a one at the same position while zeros at all others. In these cases the form of the biquadrics is analoguosly to the one of Theorem 2.74 just with permuted rows and columns. Then one can alternate the hyperplanes analogously to 2.77 while keeping the choosen p_c fixed, and finally one can translate the center point to all reachable points with translations with respect to the respective hyperplanes. This procedure covers the whole projective space and because the translations with respect to arbitrary hyperplanes at infinity are known explicitely one gets an explicit form of all biquadrics with respect to all center points for all hyperplanes (for the point not lying on that hyperplane and such that both quadrics of the pair share one polar).

Explicite Form of Biquadrics for Other Center Points in Two Dimensions

To illustrate this algorithm all biquadrics of the projective plane ($d = 2$) will be constructed. The corresponding center points and polars (lines at infinity) to start the construction process from are:

1. $p_c = \begin{pmatrix}0\\0\\1\end{pmatrix}, \ell_\infty = \begin{pmatrix}0\\0\\1\end{pmatrix}$,
2. $p_c = \begin{pmatrix}0\\1\\0\end{pmatrix}, \ell_\infty = \begin{pmatrix}0\\1\\0\end{pmatrix}$, and

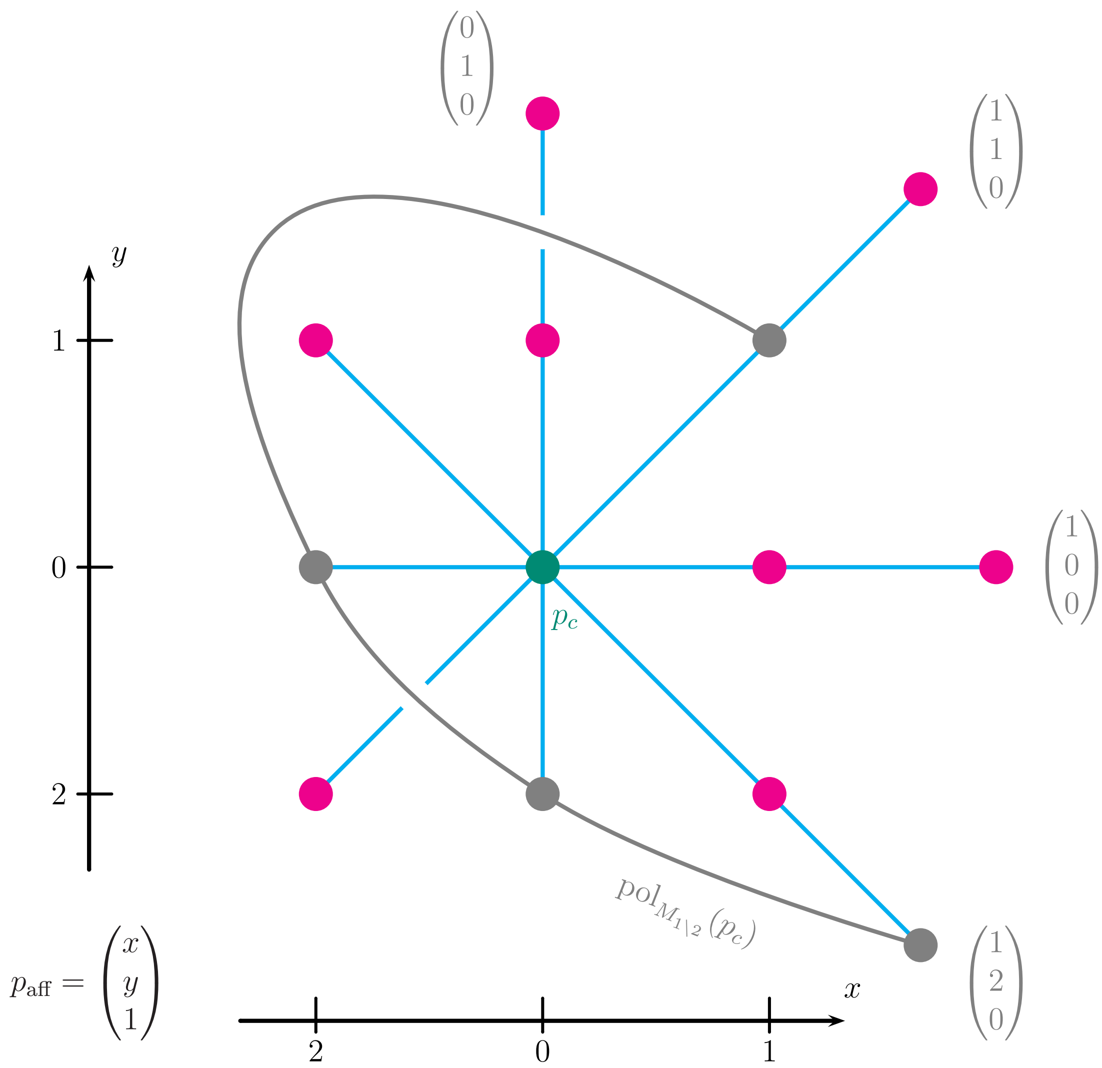


Figure 2.13: Also within the same finite projective plane $\mathbb{P}^2\mathbb{F}_3$ the same input data A, q, and $\bar{q}$ as in Example 2.75 leads for the same center point p_c (green) but a different polar $\mathrm{pol}_{M_{1\backslash 2}}(p_c)$ (gray) not to a hyperbolic biquadric: in this case it leads to the elliptic biquadric of Example 2.79 that consists of $2 \cdot (p+1) = 8$ points. All lines through p_c are secants (blue), because the biquadric does not intersect the polar.

3. $p_c = \begin{pmatrix}1\\0\\0\end{pmatrix}, \ell_\infty = \begin{pmatrix}1\\0\\0\end{pmatrix}$.

For the first paring Theorem 2.77 gives the generalization of the $\ell_\infty = (\vec{0}^t, 1)^t$ to an arbitrary ℓ_∞ of the form $\ell_\infty = (\vec{\ell}^t_\infty, 1)^t$ (as already executed above in Thereom 2.77). The projective transformations that move this biquadric to other center points are, e.g., translations and are given by:

$$\begin{aligned}
&\Pi_{(\vec{h}^t_\infty,1)^t\to(\vec{0}^t,1)^t} = \begin{pmatrix}\mathbb{1}_{d\times d} & \vec{0}\\ \vec{\ell}^t_\infty & 1\end{pmatrix}, \quad \hat{\Pi}_{(\vec{h}^t_\infty,1)^t\to(\vec{0}^t,1)^t} = \begin{pmatrix}\mathbb{1}_{d\times d} & \vec{0}\\ -\vec{\ell}^t_\infty & 1\end{pmatrix} \quad \text{and thus}\\
&T_{(\vec{\ell}^t_\infty,1)^t}(\vec{p}) = \mathbb{1} + \begin{pmatrix}\mathbb{1}_{d\times d}\\ -\vec{\ell}^t_\infty\end{pmatrix} \vec{p} \begin{pmatrix}\vec{\ell}^t_\infty\\ 1\end{pmatrix},\\
&T^{-1}_{(\vec{\ell}^t_\infty,1)^t}(\vec{p}) = T_{(\vec{\ell}^t_\infty,1)^t}(-\vec{p}) = \mathbb{1} - \begin{pmatrix}\mathbb{1}_{d\times d}\\ -\vec{\ell}^t_\infty\end{pmatrix} \vec{p} \begin{pmatrix}\vec{\ell}^t_\infty\\ 1\end{pmatrix}, \quad \text{and}\\
&T^{-t}_{(\vec{\ell}^t_\infty,1)^t}(\vec{p}) = T_{(\vec{\ell}^t_\infty,1)^t}(-\vec{p}) = \mathbb{1} - \begin{pmatrix}\vec{\ell}^t_\infty\\ 1\end{pmatrix} \vec{p}^t \begin{pmatrix}\mathbb{1}_{d\times d}\\ -\vec{\ell}_\infty\end{pmatrix}.
\end{aligned} \tag{2.3.96}$$

First Pairing: These can be used to transform both M_1 and M_2 of the biquadrics for the center point $p_c = (0, 0, 1)^t$ and $h_\infty = (\vec{h}^t_\infty, 1)^t$ (Theorem 2.77) to the new center point $(\vec{p}^t, 1 - \vec{\ell}^t_\infty \vec{p})^t$; for both $i \in \{1, 2\}$:

$$\begin{aligned}
M'_i &= T^{-t}_{(\vec{\ell}^t_\infty,1)^t}(\vec{p})\; M_i\; T^{-1}_{(\vec{\ell}^t_\infty,1)^t}(\vec{p}) = \dots\\
&= \begin{pmatrix}A_i & \vec{\ell}_\infty\\ \vec{\ell}^t_\infty & 1\end{pmatrix} - 2\cdot\mathrm{Sym}\left(\left(l_\infty \vec{p}^t \left(A_i - \vec{\ell}_\infty \vec{\ell}^t_\infty \quad \vec{0}\right)\right)\right) +\\
&\quad + \left(\vec{p}^t \left(A_i - \vec{\ell}_\infty \vec{\ell}^t_\infty\right) \vec{p}\right)\cdot \ell^t_\infty \ell_\infty,
\end{aligned} \tag{2.3.97}$$

are the matrices of the biquadric $\mathcal{B}_{(\vec{p}^t 1 - \vec{\ell}^t_\infty \vec{p})^t; (\vec{\ell}_\infty 1)^t}(A, q, \bar{q})$ for $A_1 = q\left(A - \vec{\ell}_\infty \vec{\ell}^t_\infty\right) + \vec{\ell}_\infty \vec{\ell}^t_\infty$, $A_2 = \bar{q}\left(A - \vec{\ell}_\infty \vec{\ell}^t_\infty\right) + \vec{\ell}_\infty \vec{\ell}^t_\infty$, $q \in \mathcal{Q}(\mathbb{F}_p)$, and $\bar{q} \in \bar{\mathcal{Q}}(\mathbb{F}_p)$.

Second Pairing: Analogously for the second pairing the matrices of the biquadrics $\mathcal{B}_{(0,1,0)^t,(\tilde{\ell}_\infty,1,0)^t}$ in the standard form are (in analogy to Theorem 2.74):

$$\mathcal{M}_{(0,1,0)^t;(\tilde{\ell}_\infty,1,0)^t}(A, q, \bar{q}) = \left(\begin{pmatrix} q\left(a - \tilde{\ell}^2_\infty\right) & \tilde{\ell}_\infty & qb\\ \tilde{\ell}_\infty & 1 & 0\\ qb & 0 & qc\end{pmatrix}, \begin{pmatrix} \bar{q}\left(a - \tilde{\ell}^2_\infty\right) + \tilde{\ell}_\infty & \tilde{\ell}_\infty & \bar{q}b\\ \tilde{\ell}_\infty & 1 & 0\\ \bar{q}b & 0 & \bar{q}c\end{pmatrix}\right), \tag{2.3.98}$$

where a, b, $c \in \mathbb{F}_p$, and

$$A = \begin{pmatrix} a & b\\ b & c\end{pmatrix}. \tag{2.3.99}$$

Via translations with respect to $(\tilde{\ell}_\infty, 1, 0)^t$ one obtains the matrices of all biquadric for $p_c = (p_1, 1 - p_1\tilde{\ell}, p_2)^t$:

$$M_i' = \begin{pmatrix} a_i & \tilde{\ell}_\infty & b_i \\ \tilde{\ell}_\infty & 1 & 0 \\ b_i & 0 & c_i \end{pmatrix} - 2 \cdot \mathrm{Sym}\left(\begin{pmatrix} \tilde{\ell}_\infty \\ 1 \\ 0 \end{pmatrix} \vec{p}^{\,t} \begin{pmatrix} a_i - \tilde{\ell}_\infty & 0 & b_i \\ b_i & 0 & c_i \end{pmatrix} \right) + \\ + \left(\vec{p}^{\,t} \left(A_i - \vec{\ell}_\infty \vec{\ell}_\infty^{\,t} \right) \vec{p} \right) \ell_\infty \ell_\infty^t, \tag{2.3.100}$$

for $A_1 = q\left(A - \vec{\ell}_\infty \vec{\ell}_\infty^{\,t}\right) + \vec{\ell}_\infty \vec{\ell}_\infty^{\,t}$, $A_2 = \bar{q}\left(A - \vec{\ell}_\infty \vec{\ell}_\infty^{\,t}\right) + \vec{\ell}_\infty \vec{\ell}_\infty^{\,t}$ and $a_i = (A_i)_{11}, b_i = (A_i)_{12}, c_i = (A_i)_{22}$ and $\bar{q}$ and q as usual.

Third Pairing: The remaing biquadrics are those for the $(1, \vec{0})^t$ as the polar (line at infinity):

$$\mathcal{M}_{(1,\vec{0})^t;(1,\vec{0})^t}(A, q, \bar{q}) = \left(\begin{pmatrix} 1 & \vec{0}^{\,t} \\ \vec{0} & qA \end{pmatrix}, \begin{pmatrix} 1 & \vec{0}^{\,t} \\ \vec{0} & \bar{q}A \end{pmatrix} \right). \tag{2.3.101}$$

Finally all biquadrics that are missing are those that have a center point $p_c = (1, \vec{p}^{\,t})^t$. As for the two previous cases the translations with respect to the line at infinty $\ell_\infty = (1, \vec{0}^{\,t})^t$ yield the remaing biquadrics:

$$M_i' = \begin{pmatrix} 1 & \vec{0}^{\,t} \\ \vec{0} & qA \end{pmatrix} - 2 \cdot \mathrm{Sym}\left(\begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} \vec{p}^{\,t} \begin{pmatrix} \vec{0} & A_i \end{pmatrix} \right) + \\ + \left(\vec{p}^{\,t} A_i \vec{p} \right) \ell_\infty \ell_\infty^t, \tag{2.3.102}$$

where $A_1 = qA$, $A_2 = \bar{q}A$, and q and q as usual.

Thus Equations 2.3.97, 2.3.100, and 2.3.102 show all biquadrics for all center points and polars (on which the center point does not lies) explicitely parameterized. This algorithm can be executed in arbitrary dimensions. A system is clearly visible: there is always a "$-2 \cdot$ Sym-term", where in the last factor of the matrix that is going to be symmetrized is always the hyperplane at infinity, the second factor is always the transposed affine point vector, and the third term is basically A_i but with a $\vec{0}$ column included at the position of the 1 in the hyperplane at infinity from which one started ($\ell_\infty = (0, \ldots, 0, 1, 0, \ldots, 0)^t$) and the last term is $(\vec{p}^{\,t}(A_i - \vec{\ell}_\infty \vec{\ell}_\infty^{\,t})\vec{p})\ell_\infty \ell_\infty^t$.

Example 2.80. *General Construction Method for Biquadrics in Two Dimensions*

For example for $p = 3$ *and* $d = 2$*, i.e., in* $\mathbb{P}^2\mathbb{F}_3$*, if we would like to construct a biquadric for the following data:*

$$p_c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ell_\infty = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, q = 1, \bar{q} = 2, \text{ and } A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.3.103}$$

one has to utilize the first case and calculate all ingredients and insert them into Equation 2.3.97:

$$M_1 = \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix} - \begin{pmatrix}0&0&1\\0&0&1\\0&0&0\end{pmatrix} - \begin{pmatrix}0&1&0\\0&0&0\\0&1&0\end{pmatrix} + \begin{pmatrix}0&0&0\\0&2&0\\0&0&0\end{pmatrix} = \begin{pmatrix}0&1&2\\1&0&1\\2&1&0\end{pmatrix},$$
$$M_2 = \begin{pmatrix}0&0&1\\0&1&0\\2&0&0\end{pmatrix} - \begin{pmatrix}0&0&0\\2&0&2\\0&0&0\end{pmatrix} - \begin{pmatrix}0&2&0\\0&0&0\\0&2&0\end{pmatrix} + \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix} = \begin{pmatrix}0&1&2\\1&2&1\\2&1&0\end{pmatrix}.$$

The resulting matrices form a biquadric that has been tested to be a biquadric[40] *using the computer:*

$$\mathcal{M}_{(1,1,1)^t;(0,1,0)^t}\left(\begin{pmatrix}0&1\\1&0\end{pmatrix},1,2\right) = \left(\begin{pmatrix}0&1&2\\1&0&1\\2&1&0\end{pmatrix},\begin{pmatrix}0&1&2\\1&2&1\\2&1&0\end{pmatrix}\right). \qquad (2.3.104)$$

The biquadric is shown in Figure 2.14.

Biquadric Fields

Finally in order to equip the whole finite projective geometry with a biquadric structure, one can look at a set of biquadrics, s.t., there are as many biquadrics as points in the point set P of the geometry and that each biquadric can serve as a biquadric for a different point. Such a set will be called *biquadric field* and can be seen as a bijective map of the point set into the set of all biquadrics $\mathscr{B}$:

Definition 2.81. *Biqudric Fields* $\mathcal{B}$

For a finite projective geometry $(\mathscr{P},\mathscr{H},\mathscr{I})$ *with the set of all biquadrics denoted by* $\mathscr{B}$ *a biquadric field is defined as a bijective map* $\mathcal{B}:\mathscr{P}\to\mathscr{B}$*, s.t.,*

$$\forall p\in\mathscr{P} : \mathcal{B}(p) \text{ is biquadric with center } p. \qquad (2.3.105)$$

Since there are biquadrics with more than one center point, this does not exclude that $\mathcal{B}(p)$ is also biquadric for a p' with $p'\neq p$!

$\mathcal{B}$ without any indices denotes the biquadric field, $\mathcal{B}_{p_c}$ denotes a single biquadric with the center point p_c, and $\mathcal{B}_{p_c;\mathrm{pol}(p_c)}$ denotes a biquadric with center point p_c and the corresponding polar $\mathrm{pol}(p_c)$.

If there is more than one center point the center points and polars in the indices get separated by a semi colon: $\mathcal{B}_{p_1,p_2,\ldots;\mathrm{pol}(p_1),\mathrm{pol}(p_2),\ldots}$. This notation in inherited by the matrices belonging to the biquadric.

[40] Like many more.

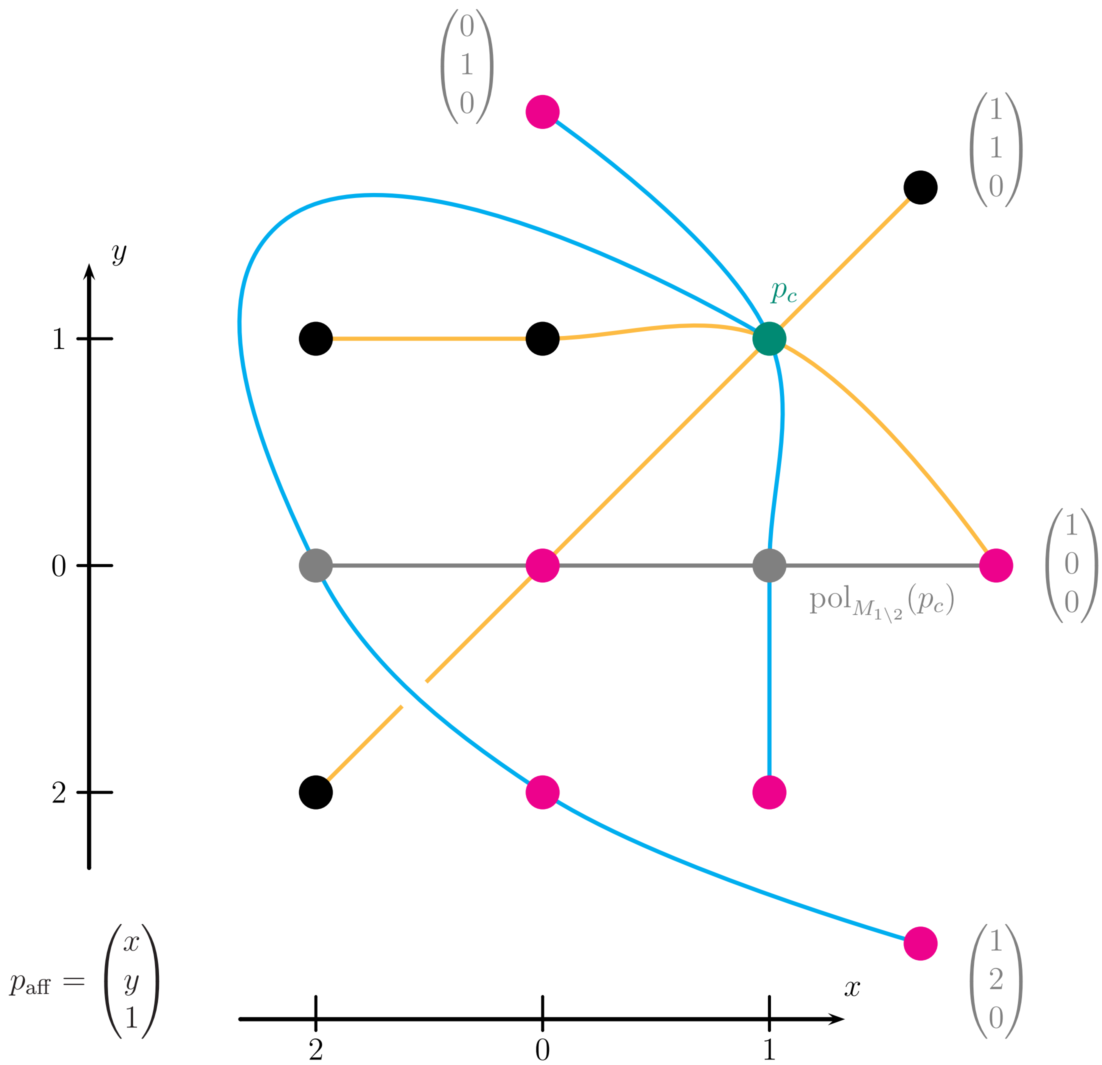


Figure 2.14: The red dots resemble the points of a biquadric in the finite projective plane $\mathbb{P}^2\mathbb{F}_3$ from Example 2.80, while the green dot resembles the center point p_c. There are only those lines shown that are incident with the center point p_c: the two blue lines are secants of the biquadric and the two yellow lines are tangents. This corresponds to the fact that the polar $\mathrm{pol}_{M_{1\setminus 2}}(p_c)$ (gray) intersects the biquadric twice: the biquaduric is hyperbolic (see Theorem 2.76) and thus there are only $2p = 6$ points that constitute the biquaudric.

An example is $\mathcal{B}_{(1,0,2)^t,(1,2,0)^t;(1,0,2)^t,(0,1,2)^t}$ with the corresponding matrices for $\mathbb{P}^2\mathbb{F}_3$:

$$\mathcal{M}_{(1,0,2)^t,(1,2,0)^t;(1,0,2)^t,(0,1,2)^t} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 2 \end{pmatrix}. \tag{2.3.106}$$

Simulations showed that there are several cases where there are not only two possible center points and two polar, there are cases with two center points and four polars as well as cases in which there are four center points and four polars. This suggests a refinement of the definition of a biquadric: *exclude multiple center points and polars.* A concise and rigorous treatment of and strong reasoning for this refinement is beyond the scope of this master's thesis.

3 Simulations of Biquadrics Fields in Finite Projective Planes

Beware of the bugs in the above code;
I have only proved it correct, not tried it.

(Donald Knuth)

In order to test how to possibly encode curvature in finite projective geometries, I wrote a C++ library for executing calculations in arbitrary dimensional finite projective geometries over Galois fields that feature additional capabilities in order to study biquadric fields. I used this library to produce random biquadric fields using the transformations developed in Section 2.2.3 and to produce uniformly distributed incidences of points in the biquadrics of a biquadric fields within an affine plane. These attempts failed, except for a brute force approach in the case of $\mathbb{P}^2\mathbb{F}_3$ that was successful. Additionally I created and counted all biquadrics in $\mathbb{P}^2\mathbb{F}_3$ and tested a manually found homogenous biquadric field that is homogenous in the whole projective space.

3.1 Motivation: Energy and Curvature

The following notion of *energy* (and its link to *curvature*) is more a working hypothesis based on an educated guess than a finalized notion: if all points are equidistant there is no curvature. "Close distance" is encoded in the biquadrics. Thus if all points lie in equally many biquadrics one might consider this state as *flat*. More or less incidences would mean that at this point there is curvature. In some limit ($p \to \infty$?) one might think that one should retrieve a theory that is very close to general relativity where the energy momentum density $T_{\mu\nu}$ is basically the same as the (*Einstein*)-*curvature* of space-time $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, where $g_{\mu\nu}$ is the *metric*, a non-degenerate, symmetric $\frac{0}{2}$-tensor, $R_{\mu\nu}$ the *Ricci-curvature tensor*, another $\frac{0}{2}$-tensor that is entirely build from the metric, and $R := R^{\mu}_{\mu}$, the *Ricci-scalar curvature* (see p. 270 et seq. in [Reb12]):

$$G_{\mu\nu} \propto T_{\mu\nu}, \tag{3.1.1}$$

for a vanishing cosmological constant.

Thus an energy defined in terms of incidence counts *might* encode curvature.

Definition 3.1. *Incidence Count* $\sharp_{\mathcal{B}}$

For a biquadric field $\mathcal{B}$ *the* *incidence count* $\sharp_{\mathcal{B}} : P \to \mathbb{N}$ *assigns to a point the number of biquadrics it lies in:*

$$\sharp_{\mathcal{B}}(p) := |\{\mathcal{B}_{p_c} \in \mathcal{B} | p \in \mathcal{B}_{p_c}\}|. \tag{3.1.2}$$

If one would define the total energy by just the sum of all the numbers of incidences all the points have with the biquadrics, then this number would simply be constant:

$$E_{\text{try}}(\mathcal{B}) := \sum_{p \in P} \sharp_{\mathcal{B}}(p) = \langle \sharp_{\mathcal{B}} \rangle \cdot |\mathcal{B}| = \text{ const.} \tag{3.1.3}$$

The mean value is $\langle \sharp_{\mathcal{B}} \rangle = 2p$ for a purely hyperbolic biquadric field and $\langle \sharp_{\mathcal{B}} \rangle = 2(p+1)$ for one that only consists of elliptic biquadrics. For all other mixed compositions of hyperbolic and elliptic biquadrics $\mu(\sharp_{\mathcal{B}})$ lies inbetween these two values. But for a fixed composition (how many biquadrics are hyperbolic and how many are elliptic) this value is constant and therefore cannot detect how the state deviates from a flat state. Thus in order to introduce a notion of energy that is sensitive to distinguish the flat state from curved ones, one at least needs an energy that adds the squared values of incidence counts (or some non-linear function that treats all incidence counts of the various points equally). But this is the most simple energy functional:

Definition 3.2. *Energy E of a Biquadric Field* $\mathcal{B}$

$$E(\mathcal{B}) := \sum_{p \in P} \sharp_{\mathcal{B}}(p)^2. \tag{3.1.4}$$

This energy is basically the *variance* $\langle \sharp_{\mathcal{B}}^2 \rangle - \langle \sharp_{\mathcal{B}} \rangle^2$ of the distribution of incidences of points in biquadrics up to a multiplicative factor and a constant offset. Now one can employ this notion and the underlying ideas of how to encode curvature in finite projective geometries and try to find a ground state (uniformely distributed incidence counts) that would correspond to a flat spacetime, look at random biquadric fields, or examine properties of this new notion literally in an explorative sense. But the immense amount of calculations needed to execute these tasks neccesitates the use of a computer.

3.2 A Brief Description of the C++ Library

A functional framework that allows one to tackle much more questions then whether this energy functional is an appropriate notion in order to encode curvature (within finite projective geometries) is a library. I decided to write a C++ library due to the good performance because of its relatively low-level machine-orientedness.

It has the working title "`libgalois`" and serves with all notions of *scalars*, *points*, *lines*, *hyperplanes*, *quadrics*, *biquadrics*, *endomorphisms* and more in order to calculate within $\mathbb{P}^d\mathbb{F}_p$ in principle for an arbitrary dimension d and an arbitrary prime number p[1]. Additionally there are many operations to combine the geometric (and standard linear algebraic) objects sensefully.

Everything is designed with the "tensor picture" of projective geometries in mind. For example a point is a so-called `galoisOneZeroTensor` that can only be acted upon by appropriate tensors like their dual hyperplanes, a `galoisZeroOneTensor`, e.g., in this case yielding the value of their scalar product.

An endomorphism is represented by a `galoisOneOneTensor`, while a `galoisZeroTwo Tensor` represents the quadrics that either map a single point (`galoisOneZeroTensor`) onto its polar, a hyperplane (`galoisZeroOneTensor`) or two points into $\mathbb{F}_p$ (a `galois Scalar`).

All tensors contain information only up to homogenity and thus all the comparison operators are defined up to homogenity, i.e., only tensors of identical tensor rank are able to be compared (thus no lines with points) and in order to get compared they get normed with respect to the very same element (this is always possible without loss of generality, because if they cannot be normed with respect to the same element, one of them has to have a zero component for the same index for which the other does have a non-vanishing component and one immideatly knows that they cannot be equal).

The tensor classes are inherited from the classes `galoisVector` and `galoisMatrix`, but both of them are not availabe directly and only all derived objects (the tensors and biquadrics) are usable in order to prevent one from executing operations that mix objects that have nothing to do with each other and thus prevent one from executing senseless calculations[2].

The basic geometrical objects are supplemented with a class, `finiteProjective Geometry`, that contains methods that apply to all objects like a method for finding all objects that are incident with a given objects, e.g., all lines through a point using the explicite parameterization from Equation 2.48. Another method tests whether a point is center point of a biquadric.

The syntax of how to treat these objects is encoded in the operators that make the object act on each other. This syntax tries to mimic the way one would write down terms on paper[3].

[1] The Galois fields with respect to p^s, for p a prime number and $s \in \mathbb{N}\backslash\{0\}$ might be added soon in case they are neccessary. Until now $s = 1$ is used, because there the multiplication is just the ordinary multiplication of intergers to the modulus of the prime number p.

[2] If one nevertheless needs to compare technically incommensurale objects, one can force those comparisons (for example the componenent-wise comparison of a point and a hyperplane); but in general something like this cannot happen by accident.

[3] An exception are transpose signs.

For example for `p`, `q` `galoisOneZeroTensor`, `h` `galoisZeroOneTensor`, and `M` `galoisZeroTwoTensor` one can evaluate "`pMp`", one could not evaluate "`hM`", but one could assign "`h = Mp`". It is also possible to let them act on each other in the "function(argument) picture": `pMp = (M(p))(p)`, expressing that "first" `M` acts on its argument `p` resulting in the polar that in turn acts on the point `p` that yields a GaloisScalar (that signals that `p` lies on $Q_{\mathtt{M}}$ if it is equal to zero).

3.3 All Biquadrics in $\mathbb{P}^2\mathbb{F}_3$

In order to examine this new notion of a `biquadric` or how to encode curvature with biquadric fields one can try to produce all biquadric in $\mathbb{P}^d\mathbb{F}_p$ (they are finitely many and hence this is possible in principle) and examine their forms, count them in total, count how many biquadrics are there for a fixed center point, and so forth. This process can lead to suitable conjectures for smaller prime numbers that could be tried to be proven analytically for higher prime numbers later on.

The Prequel Case of $\mathbb{P}^2\mathbb{F}_2$

The smallest finite (non-degenerate) projective plane is the yet mentioned Fano plane (see Figure 2.2(b)). The plane itself is non-degenerate (it fulfills axiom **P3**), but the notion of a biquadric is somehow degenerate here, because there is only one square number (1) and no number without a square root (see the general form of biquadrics with one center point in Theorem 2.77)). Furhtermore, there is no $1/2 = 2^{-1}$ in $\mathbb{P}^2\mathbb{F}_2$, which makes the notions of symmetry and anti-symmetry obsolete. Therefore the smallest finite projective geometry of interest for investigating biquadrics is $\mathbb{P}^2\mathbb{F}_3$.

The Case of $\mathbb{P}^2\mathbb{F}_3$ Itself

I used the `libgalois` library to produce all biquadrics (for all center points with all possible polars) by:

- generating all quadrics by considering all invertible matrices,
- gather them into all pairs of quadrics (that do not have to be biquadrics!),
- test for all points whether there are two intersections with the biquadric for all lines through that point, and
- thus identify the biquadrics within all pairs of non-degenerate quadrics.

The following observations where possible:

- There are 5265 ($= 5 \cdot 13 \cdot 3^4$) biquadrics in total,

- 1053 ($= 13 \cdot 3^4$) of these have <u>one</u> center point and <u>one</u> polar,
- 4212 ($= 4 \cdot 1053 = 4 \cdot 13 \cdot 3^4$) of these have <u>two</u> center points and either <u>two</u> or <u>four</u> polars respectively,
- hence there are <u>no</u> other cases (like, e.g., three center points and six polars),
- in the cases of a <u>single</u> center point there are always 81 ($= 3^4$) biquadrics for a fixed center point but variable polar,
- also in the case of a <u>single</u> center point there are 9 ($= 3^2$) biquadrics for a fixed center point and a fixed polar,
- in the case of <u>two</u> center points there are 648 ($= 12 \cdot 54$) cases for one of the two center points fixed and the other center point as well as all polars kept variable, and
- also in the case of <u>two</u> center points there are 54 cases for both center points fixed while the polars are still variable.

There are 9 biquadrics per point and polar because there are 18 different regular matrices A that can be inserted in the form given in Theorem 2.74, but there is only one number without a square root (2) in $\mathbb{F}_3$ and thus half of the 18 cases have a unique counterpart ($2A$) that leads to a biquadric; hence there are 9 biquadrics per fixed point and fixed line at infinity. The nine corresponding matrices A are

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&2\end{pmatrix}, \begin{pmatrix}1&2\\2&1\end{pmatrix}, \begin{pmatrix}1&1\\1&2\end{pmatrix}, \begin{pmatrix}1&2\\2&2\end{pmatrix}, \begin{pmatrix}1&1\\1&0\end{pmatrix}, \begin{pmatrix}1&2\\2&0\end{pmatrix}, \begin{pmatrix}0&1\\1&1\end{pmatrix}, \text{ and } \begin{pmatrix}0&2\\2&1\end{pmatrix}.$$

Furthermore there are $p^2 = 3^2 = 9$ different polars on which the center point does not lie and hence there are $9 \cdot 9 = 81$ biquadrics per fixed center but for an arbitrary polar on which this center point does not lie. Finally there are 13 points in $\mathbb{P}^2\mathbb{F}_3$ leading to $81 \cdot 13 = 1053$ biquadrics with only one center point. This consideration is in total agreement with the results obtained above by producing all biquadrics and counting the respective cases.

There are 78 possibilities to select two different center points out of the set of 13 points in $\mathbb{P}^2\mathbb{F}_3$. So far there is no explicitely parameterized form of biquadrics with respect to two center points and polars and hence there is no "theory" to derive that there are 54 cases for two fixed center points; but at least it is consistent with the rest of the counts: there are 12 possibilities to choose the second center point from, in case one center point is already fixed. Thus there have to be $^{648}/_{12} = 54$ biquadrics for two fixed center points. Finally these 54 biquadrics per pair of center points times the number of such pairs should yield the total amount of biquadrics with two center points and indeed it does: $54 \cdot 78 = 4212$.

Hence all possible pairs of points are able to serve as center points of biquadrics with two center points.

Sequel Cases $\mathbb{P}^2\mathbb{F}_p$ for $p > 3$

Higher prime numbers lead to much longer calculation times, s.t., no complete list of biquadrics for higher prime numbers has been created so far ($p = 5$ and $p = 7$ are still possible within some days or months respectively and thus the case of $p = 5$ is going to be examined soon). But it is possible to run the program for a little while and examine the results *so far*:

- There are also biquadrics with one center point and one polar and two center points and either two or four polars,
- there also are biquadrics with four center points and four polars, and
- there are no biquadrics with three center points yet.

Both an explicite form of biquadrics with more than one center point and the full list of biquadrics for $p = 5$ would clarify this circumstance at least a little more. One cannot say whether there are not cases for which there are differently many center points then polars.

These ambiguities that occur if one allows for more than one center point seem unnatural and suggest to exclude this possibility in the definition of a biquadric. Another reason for this uniqueness of the center point is: once there is a unique center point one can easily interpret the unique polar as a hyperplane at infinity with the "nice" property that biquadric points lie symmetrical with respect to the dehomogenized center point in the resulting affine plane (see Lemma 2.78):

$$\forall p_c \in P, h \in H, h_\infty^t p_c \neq 0 : \quad \forall p \in P, p \in \mathscr{B}_{p_c;h_\infty} : T_{h_\infty}(2\mathfrak{D}_{h_\infty}(p_c - p))p \in \mathscr{B}_{p_c;h_\infty}. \tag{3.3.5}$$

In the case of multiple polars it is not clear with respect to *what* line (or hyperplane) at infinity this translation takes place and with respect to both at the same time a translation is not defined.

3.4 Random Biquadric Fields

The transformation matrices developed in Section 2.2.3 are mapping a point onto another (or the same) point while mapping a hyperplane onto another (or the same) hyperplane. One can utilize this to map the center point of a biquadric onto a new center point while also mapping its polar onto a new polar. There are stil degrees of freedom that can be choosen randomly. This leads to the following construction of random biquadric fields:

- Select a random biquadric $\mathscr{B}_{(0,0,1)^t;(0,0,1)^t}$ with respect to the center point $p_c = (0,0,1)^t$ and the polar $\text{pol}(p_c) = (0,0,1)^t$, e.g., with matrices

$$\mathcal{M}_{(0,0,1)^t;(0,0,1)^t} = \left(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \bar{q} \end{pmatrix} \right) \tag{3.4.6}$$

  and $\bar{q} \in \mathscr{Q}(\mathbb{F}_p)$,

- select for all points in $\mathscr{P}$ randomly a polar, randomly two points on this polar, and two scaling factors ($\in \mathbb{F}_p \backslash \{0\}$) in order to scale the first two columns for constructing the transformation matrices Π according to Equation 2.2.69[4],

- transform the matrices of the initial biquadric $\mathcal{M}_{(0,0,1)^t;(0,0,1)^t}$ to all new center points with their according polars, and

- count the incidences of all points and calculate $E(\mathcal{B})$.

This procedure consitutes *one run*. Many of these can be executed and afterwards added to smooth out "noise" within the single runs. For all tested prime numbers (all from 3 up to 31 and some higher prime numbers, e.g., 73, 83, 101, and 199) the histogram of the incidence counts $\sharp_{\mathcal{B}}$ seems to be distributed approximately Gaussian. The mean value of the distribution is — as one would expect – inbetween $2p$ and $2 \cdot (p+1)$, because in the hyperbolic case there are $2p$ points that form a quadric and in the elliptic case there are $2 \cdot (p+1)$. The mean value is slighlty shifted to $2p$ in all cases corresponding to the existence of more hyperbolic than elliptic quadrics. For higher prime numbers the shift suggests that there are *twice* as much biquadrics with $2p$ points as biquadrics with $2 \cdot (p+1)$ points. For a relatively high prime number ($p = 73$) the same results are found and are shown in Figure 3.3. In Figure 3.4 are all the histograms after 1000 runs for all prime numbers between 3 and 19. Figures 3.1 and 3.2 show the histograms after 1000 runs for the prime numbers $p = 3$ and $p = 19$, as well as one heatmap each.

That all random biquadric fields behave like expected is a good consistency check for both the transformations and the library.

[4]The commonly used Mersenne-Twister MT 19937 had also been utilized to select the indices in the corresponding point sets uniformely. Mind that this construction method is equvivalent to write the new center point into the last column of a transformation matrix while selecting 6 random values to fill the first two columns of the transformation matrix (and checking whether this matrix is an appropriate transformation by testing non-degeneracy).

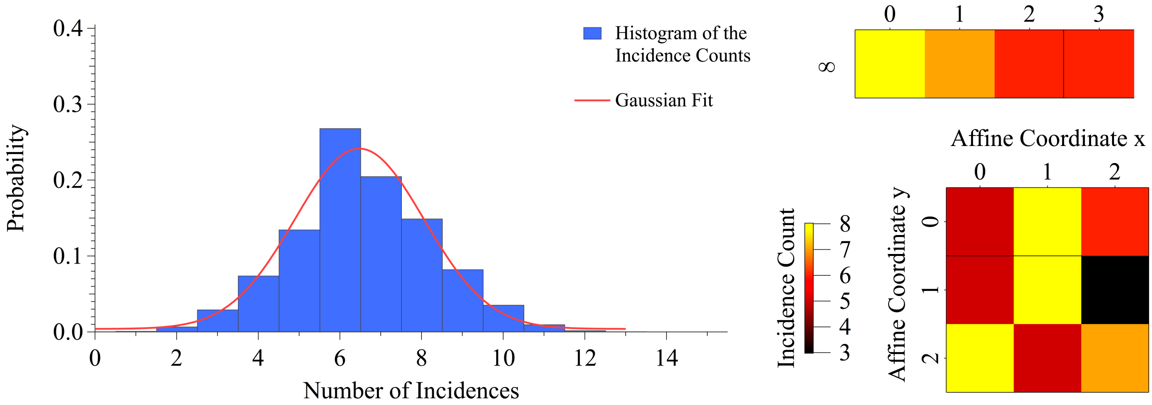


Figure 3.1: On the left there is the histogram of the incidence counts $\sharp_{\mathcal{B}}$ for $p = 3$ (the sum of 1000 histograms is displayed). The mean value 6.5 ± 0.09 and the standard deviation 3.1 ± 0.17. On the right there is the heatmap of one run, i.e., the incidence counts $\sharp_{\mathcal{B}}$ for $p = 3$ for a random biquadric field $\mathcal{B}$ that had been produced using the method described in Section 2.2.3. In the top on the right hand side there are the points of the line at infinity and x and y refer to the affine coordinates within the affine plane $(\vec{p}^{\,t} = (x, y)^t)$.

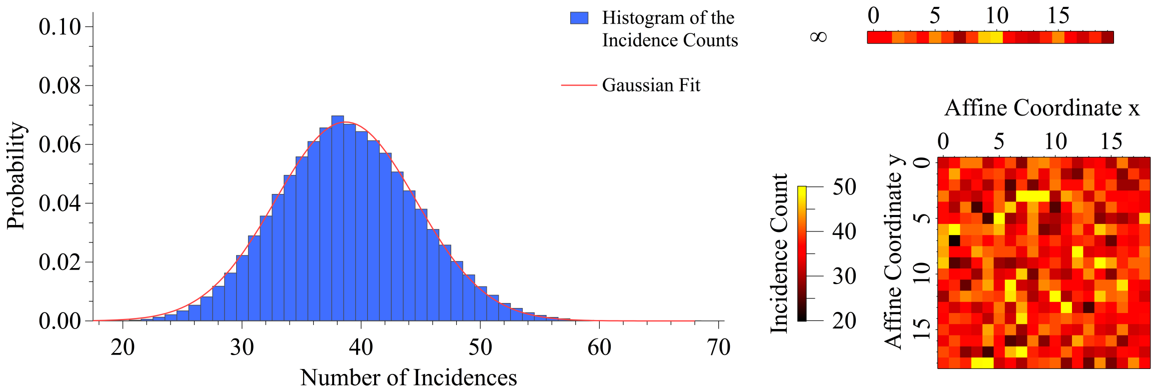


Figure 3.2: On the left there is the histogram of the incidence counts $\sharp_{\mathcal{B}}$ for $p = 19$ (the sum of 1000 histograms is displayed). The mean value 38.7 ± 0.03 and the standard deviation 11.8 ± 0.04. One can see that the slope of the Gaussian is slightly shifted to the right with respect to the actual slope of the histogram corresponding to the existence of more hyperbolic than elliptic biquadrics and that the distribtion is not Gaussian. It is rather a weighted sum of two Gaussian distributions corresponding to the hyperbolic and elliptic biquadrics. On the right there is one heatmap for $p = 19$ for a random biquadric field $\mathcal{B}$ that had also been produced using the method described in Section 2.2.3.

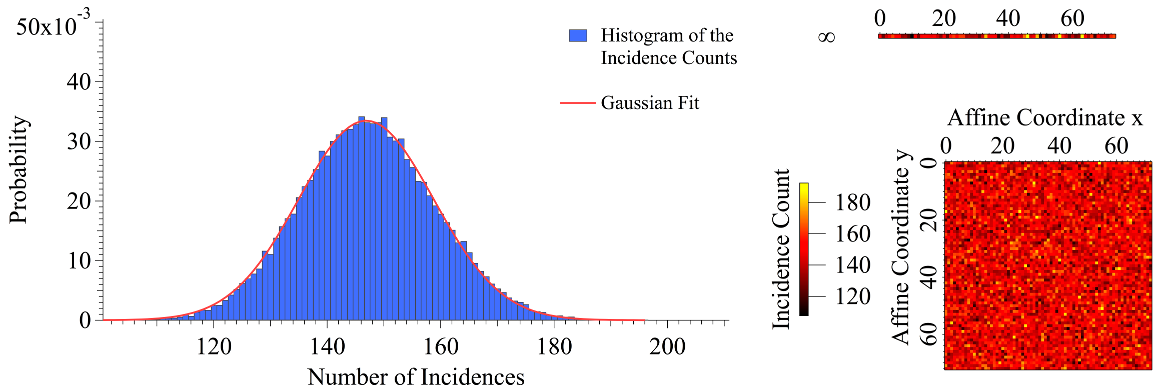


Figure 3.3: On the left there is the histogram of the incidence counts $\sharp_{\mathcal{B}}$ for $p = 73$ (only the sum of 10 histograms is displayed). The mean value 146.9 ± 0.04 and the standard deviation 23.8 ± 0.06. Here only 10 runs have been added, because of the long calculation time.

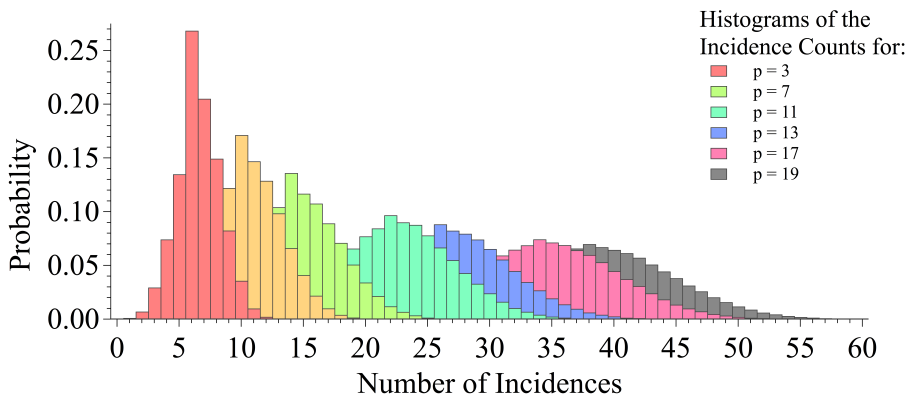


Figure 3.4: All histograms of incidences after 1000 runs for all prime numbers between 3 and 19.

3.5 Homogenous Biquadric Fields

3.5.1 The Search for a Flat Spacetime

In order to produce a flat ground state we first looked at the problem of producing a flat affine plane while the line at infinity does not have to be flat. The idea had been that translations (without loss of generality with respect to $l_\infty = (0,0,1)^t$) of a given biquadric — such that its center is mapped onto all affine points — yields a flat distribution of incidences within the affine plane *so far*[5]. But the remaining $p+1$ biquadrics with center points on the line at infinity are yet missing. Some points of these biquadrics lie in the affine plane (that had been flat "after" the p^2 translations) and thus are able to destroy the flat state.

Thus several transformations in order to map a biquadric $\mathcal{B}_{(0,0,1)^t;(0,0,1)^t}$ with center $(0,0,1)^t$ and polar $(0,0,1)^t$ onto biquadrics with center points on the line at infinity in such a way that the incidences that are thus distributed over the affine plane keep it flat. Numerous attempts had been tried. In the following there are just some examplary cases. All attempts that are presented below started with the same initial biquadric as for the random biquadric fields (Equation 3.4.6). Both matrices belonging to the initial biquadric of these attempts have the form

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \kappa \end{pmatrix}, \tag{3.5.7}$$

for κ once is equal to 1 (a number that posses square root) and once to a number without a square root (a $\in Q(\mathbb{F}_p)$. On this biquadric there are the points:

$$Q_M = \left\{ \begin{pmatrix} \mu \\ -\frac{\kappa}{2\mu} \\ 1 \end{pmatrix} \mid \lambda \in \mathbb{F}_p \backslash \{0\} \right\} \cup \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}. \tag{3.5.8}$$

The two points of them that lie on the line at infinity $l_\infty = (0,0,1)^t$ are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{3.5.9}$$

$p+1$ of the points on the initial line at infinity that are not center points yet, had been tried to be reached, e.g., with this series of transformations for $\lambda \in \mathbb{F}_p \backslash \{0\}$

$$\zeta(\lambda) = \begin{pmatrix} 1 & -\frac{1}{\lambda} & \lambda \\ -\lambda & 1 & 1 \\ -\frac{1}{\lambda} & -1 & 0 \end{pmatrix} \tag{3.5.10}$$

[5]The flatness of the affine plane after the p^2 translations as a partial result has been tested positively as another consistency check

or also for $\lambda \in \mathbb{F}_p \backslash \{0\}$ with

$$\eta(\lambda) = \begin{pmatrix} 1 & 0 & \lambda \\ 0 & 1 & 1 \\ -\lambda & -1 & 0 \end{pmatrix}, \tag{3.5.11}$$

while the target center points

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{3.5.12}$$

had been reached by

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{3.5.13}$$

The heatmaps of the resulting biquadric fields are shown in Figures 3.5 and 3.6 and neither of them is really flat. Both series of transformations (and a lot others that had been tried) had in mind to map the initial line at infinity successively onto parallel lines that cover the initial affine plane uniformely in order to spread the according incidences also uniformely or to map in half of the cases $l = (0, 1, 0)^t$ onto the initial line at infinity $(0, 0, 1)^t$ and in the other half $l = (1, 0, 0)^t$ onto the intial line at infinity because they are the tangents to the initial biquadric and contain no incidences in the intitial affine plane, such that these "lacks of incidence counts" are shifted to the initial line at infinity.

The attempt that approached flatness most accurately had also been using translations in the affine plane to reach the first p^2 points, the matrices

$$\theta(\lambda) = \begin{pmatrix} 0 & 1 & \lambda \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{3.5.14}$$

in order to transform to the points of the form $p_c = (\lambda, 1, 0)^t$ for $\lambda \in \mathbb{F}_p \backslash \{0\}$ and

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.5.15}$$

for reaching the last missing point $p_c = (1, 0, 0)^t$ on the line at infinity. The result is "pretty flat" in the sense that there are only twice as many incidences that one biquadric could serve with misplaced and this seems to be independent of the prime number p. Thus for $p \to \infty$ the distribution becomes more and more homogenous. The resulting heatmap and histogram are shown in Figure 3.7.

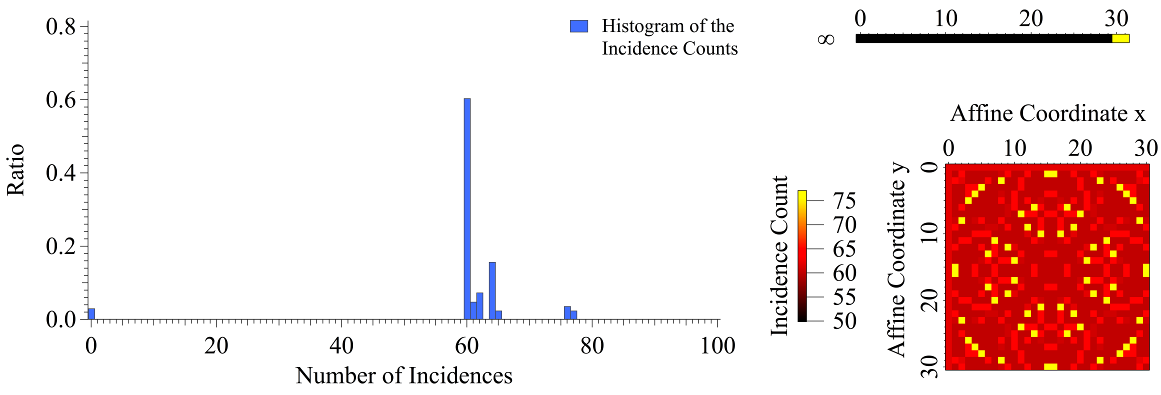


Figure 3.5: On the left there is the histogram for the transformation series $\zeta(\lambda)$ of Equations 3.5.10 and 3.5.13. It shows clearly that the produced biquadric field is not flat. This attempt had been executed for several prime numbers (of which $p = 31$ is shown here). One can also see that only the translations populated the affine plane in total. The additional incidences that lead to a non-flat distribution do not affect some lines of the affine plane, e.g., the two lines $(1,0,0)^t$ and $(0,1,0)^t$ that already play a special role formally (in the left and on the top of the affine plane). On the right there is the heatmap for this series of transformations that also indicates that there are too many incidences that are misplaced with respect to a flat distribution.

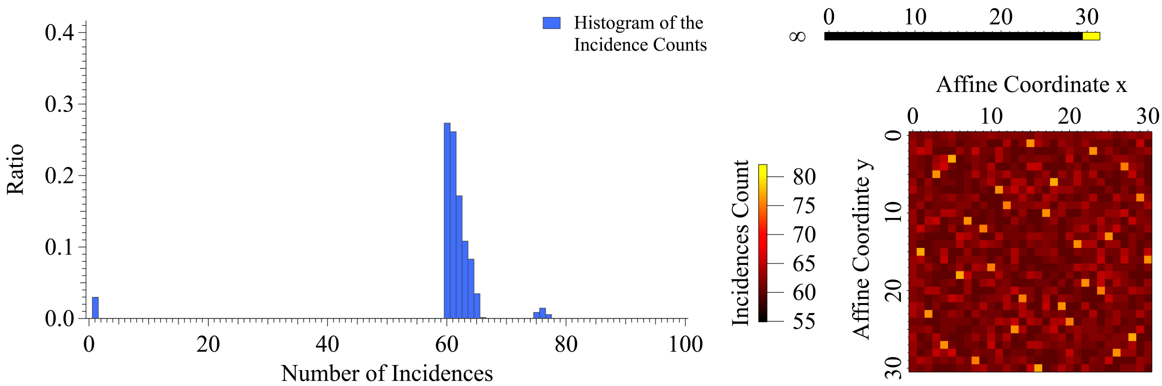


Figure 3.6: On the left there is the histogram for the transformation series $\eta(\lambda)$ presented in Equation 3.5.11 and 3.5.13 the resulting distribution is also not flat, but this time the whole affine plane gets "sprinkled" with incidences and the two lines $(1,0,0)^t$ and $(0,1,0)^t$ are affected as well. On the right there is the histogram belonging to this transformation series $\eta(\lambda)$ showing that this attempt leads to a state that is even less flat. The peak is not as high as for 3.5 and the slope expands to the right.

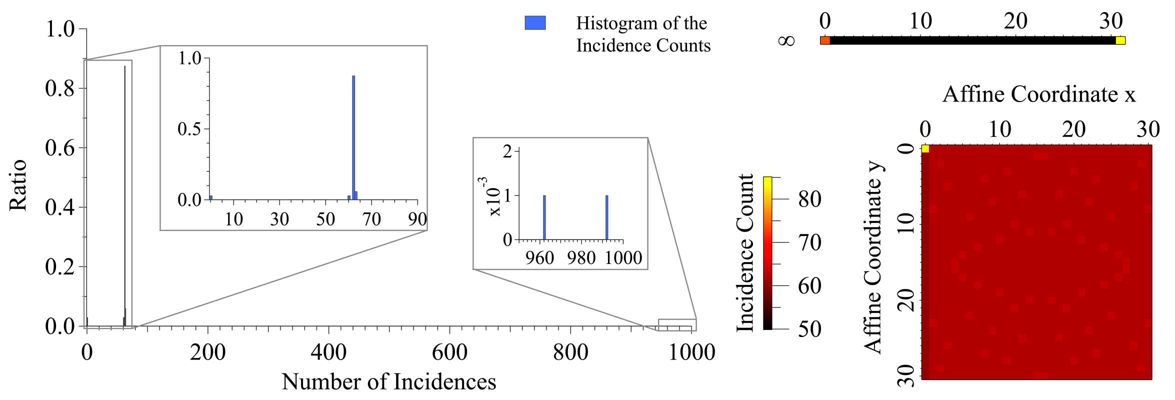


Figure 3.7: The best but still not flat distrubtion had been produced by the transformation series $\theta(\lambda)$ shown in Equations 3.5.14 and 3.5.15. But still there are some incidences misplaced and the line $(1,0,0)^t$ is again not reached by the additional incidences that are distributed in order to produce biquadrics centered around points on the line at infinity. On the left there is the histogram for this attempt. The line $(1,0,0)^t$ is "underpopulated" and corresponds to the small peak on the left of the main peak (visible in the left inset).

3.5.2 Homogenous Biquadric Fields in Affine Planes of $\mathbb{P}^2\mathbb{F}_3$

The flat affine plane could nevertheless be realized by trying all transformations to the line at infinity "after" the affine plane had been covered by the usual p^2 many translations for the prime number $p = 3$, i.e., trying *all* transformations that map the standard center point to all of the points on the line at infinity. There are realizations of states that feature a homogenous distribution of incidences in the biquadrics:

- There is a flat affine plane with two different incidence counts for the line at infinity. Two of them have a very low incidence count and in the two others have a very high incidence count. The corresponding heatmap as well as the histogram is shown in Figure 3.8.
- There is another flat affine plane for which the line at infinity has more structure: two very little but differently populated points and two very high but again differently populated points. The heatmap and the histogram are shown in Figure 3.9.
- There is state that has the same energy, but is not flat. This happens due to the circumstace that "flat" in this context means "flat within the affine plane" and not "flat within the whole projective space". Thus the energy of such flat states is not minimal and there can be other states with the same energy that do not feauture a homogenous distribution of incidences in biquadrics. Such a state that is not flat, but has the same energy is shown in Figure 3.8.

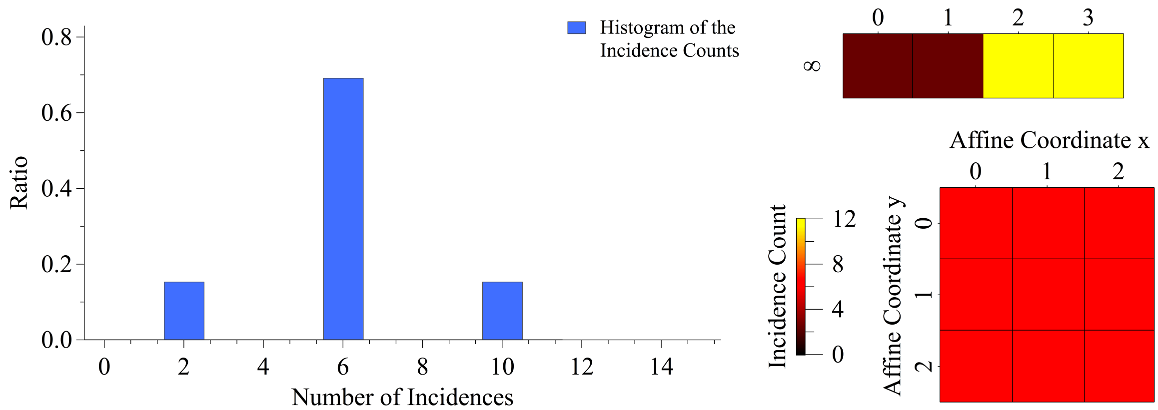


Figure 3.8: On the left there is the histogram belonging to the case where there are two points with a very low incidence count and two with a very high incidence count on the line at infinity and all 9 points of the affine plane have the same incidence count $\sharp_{\mathfrak{B}} = 6 = 2p$. On the right there is the heatmap for this case ($p = 3$).

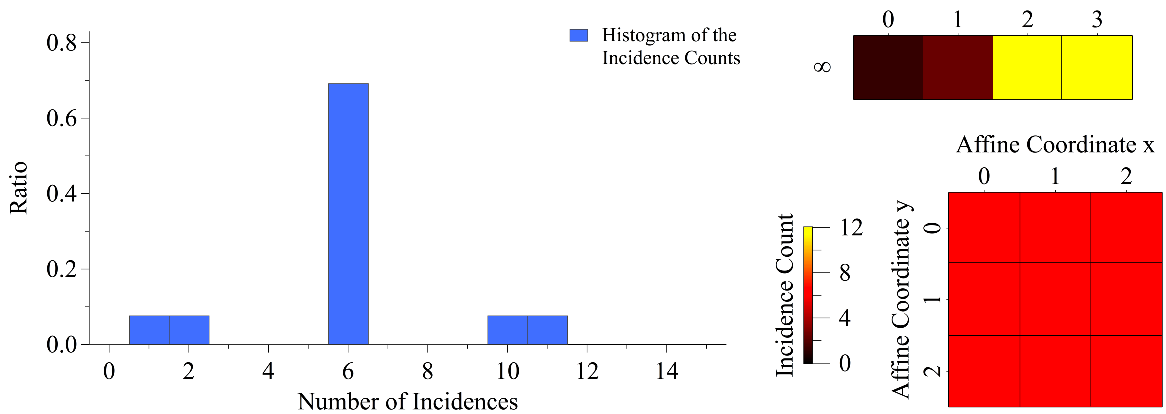


Figure 3.9: On the left there is the histogram belonging to the case that there are four different incidence counts on the line at infinity infinity. The huge peak is produced by the incidence counts of the flat affine plane. On the right there is the heatmap for this case. The affine plane is flat, but one can see the structure on the line at infinity. This flat state corresponds also to a biquadric field over $\mathbb{P}^2\mathbb{F}_3$.

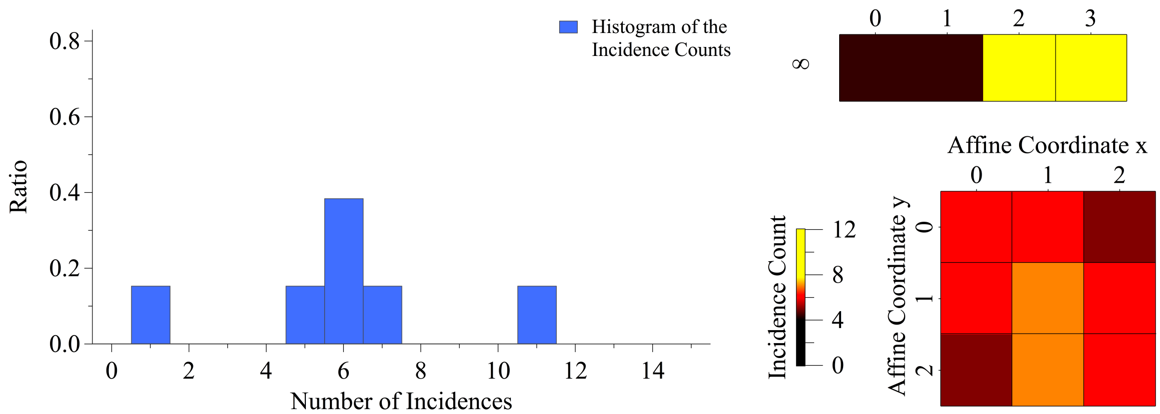


Figure 3.10: These figures show the histogram (left) and the heatmap (right) of a state (for $p = 3$) that is not flat but for which the energy functional $E(\mathcal{B})$ restricted to the affine plane (the sum in Definition 3.2 is only to be taken over the affine plane) has the same value as for the flat state. This is only possible due to the restriction. It allows to have less incidences in the affine plane in total than for the flat states. Otherwise it would not be possible for the quadratic functional to be minimized except for a homogenous distribution of incidences.

All ground states, espaccially the flat one, are degenerate, because all affinities map the affine plane onto itself and hence do not change the incidence counts that occur in the affine plane.

For example the following four transformation matrices in order to reach the center points that lie on the initial line at infinity lead to the flat state with the two different incidence counts on the line at infinity:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 1 & 2 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \tag{3.5.16}$$

while all affine points are reached by the translations (see Theorem 2.69). The according heatmap and histogram are shown in Figure 3.8.

An example for the transformation matrices leading to the flat state, for which the line at infinity features four different incidence counts is shown in Figure 3.9 and the accroding matrices are

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 1 & 2 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \tag{3.5.17}$$

while again the affine plane is covered by translations.

The state that is not flat is, e.g., produced by the following four matrices for reaching the center points that lie on the initial line at infinity (see Figure 3.10):

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}. \tag{3.5.18}$$

3.5.3 Homogenous Biquadric Fields in the Whole Projective Space $\mathbb{P}^2\mathbb{F}_3$

The symmetry breaking of selecting a line at infinity, e.g., $(0, 0, 1)^t$ as above, is somehow unnatural and there is no obvious physical reason so far that inspires one to do so. Thus I tried to find a ground state (on paper by hand) that is flat in the whole projective geometry, found one and double checked it using the library.

Figure 2.6 shows the analogon of the Fano plane for $p = 3$ and all lines. I executed the follwoing steps for each point $p \in \mathscr{P}$:

- identify all lines that are incident with that particular point,
- select two points on each line that has not been choosen as a biquadric point more than or 8 times, and
- collect these points that form a biquadric and solve the linear equation in the components of the matrices belonging to quadrics ($p^t M p = 0$ yields a row of the linear equation for all points p on Q_M).

The resulting biquadrics are elliptic and thus have $2 \cdot (p + 1) = 8$ points each. The completely flat state is shown in Figure 3.11. The list of points of all biquadrics as well as the matrices together with the corresponding polars (see Table 3.1).

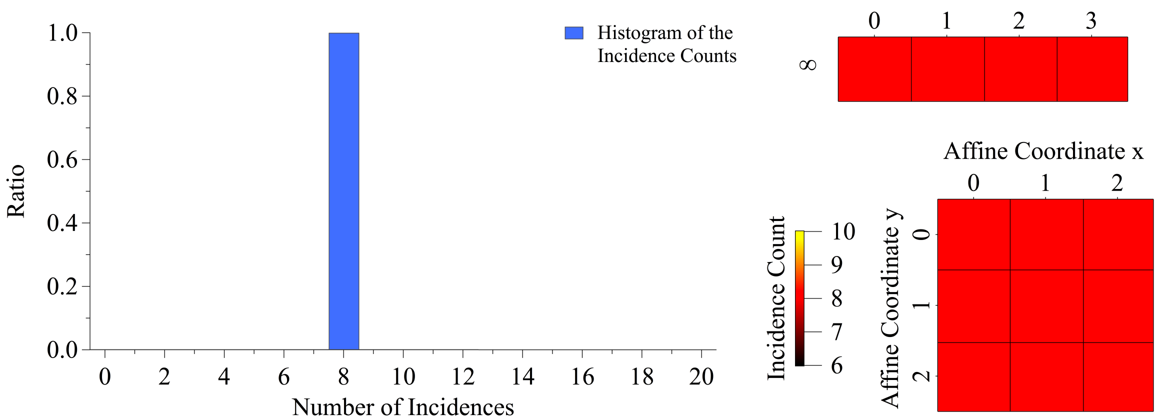


Figure 3.11: On the left there is the histogram of incidences for the manually found flat biquadric field (that is flat in the whole projective space). There is a single peak at 8 because all 13 points ($p = 3$) are incident with $2 \cdot (p + 1) = 8$ biquadrics.

All projective transformations preserve incidence and thus all projective transformations preserve this homogenous state. This flat state nevertheless involves biquadrics that feature the ambiguity of being able to serve as a biquadrics for more than one center point. It remains as an open question whether there is a biquadric field that leads to a homogenous distribution of incidences but at the same time only involves biquadrics with a unique center point.

p_c	$\mathcal{B} = (Q_{M_1}, Q_{M_2})$	$\mathcal{M} = (M_1, M_2)$	$\mathrm{pol}_{M_{1\setminus 2}}(p_c)$
$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\2\\1\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}; \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$
$\begin{pmatrix}1\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}; \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&2&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$
$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}; \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}$	$\begin{pmatrix}2&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$
$\begin{pmatrix}2\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}; \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}0\\2\\1\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&2&0\\1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&2&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}2\\0\\1\end{pmatrix}$
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$\begin{pmatrix}1\\2\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}; \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\2\\1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&0&0\\0&0&2\end{pmatrix}, \begin{pmatrix}1&0&2\\0&1&1\\2&1&1\end{pmatrix}$	$\begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\2\end{pmatrix}$

Figure 3.12: The table of all biquadrics of the biquadric field that is flat in the whole projective plane. The underlying matrices M_1 and M_2, the point sets $\mathcal{B}$, and the corresponding polars $\mathrm{pol}_{M_{1\setminus 2}}(p_c)$ are assigned to their respective center points p_c.

4 Summary and Open Questions

> The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve.
>
> (Eugene Wigner)

In order to practice physics within finite projective geometries the notion of *one* quadric does not suffice to encode length in all directions and thus a pair of quadrics, a *biquadric*, has to be employed. This notion is an entirely new concept and not much is known about it. Hence the main scope of this master's thesis had been to broaden the knowledge about this new notion. I found an explicetly parameterized matrix form of the embedding of an affine space into a projective space as well as of affine transformations represented be projectivities both with respect to an arbitrary hyperplane at infinity. Furthermore there are two main achievements of this master's thesis: the generalization of the explicitely parameterized form of biquadrics from the specific center point $p_c = (0, 0, 1)^t$ and the specific polar $\text{pol}_{p_c} = (0, 0, 1)^t$ in two dimensions to arbitrary center points and arbitrary polar hyperplanes in arbitrary dimensions. The coding of a library that enables one to calculate within finite projective geometries and first applications of it are the second achievement. Besides positive consistency checks I simulated random biquadric fields in the projective plane, found two different types of flat states (only within an affine plane and within the whole projective plane), and I produced a systematic list of all biquadrics for $\mathbb{P}^2\mathbb{F}_3$ and counted cases. The counts were in agreement with the theoretical considerations. There appeared cases where a biquadric can serve as a biquadric for several center points which lead to a suggestion of a refinement of the definition of a biquadric (namely excluding these ambiguous cases).

There remain a lot of open questions and tasks to encounter. Some of them are:

- Is there a strong reason to restrict the definition of a biquadric to cases where the center point is unique? Is there even a real physical reason?
- Produce all biquadrics for $p = 5$ and maybe $p = 7$; both should still be possible within some days or weeks of calculation and even $p = 11$ or $p = 13$ in case of a revision of the code in order to optimize the performance.
- Investigate a "continuum limit" in order to reproduce at least a spacetime that is close to a smooth structure. A promising way therefore is to search for a suitable phase transition that features scale invariance at a certain critical point.

- The general form of biquadrics admits possibly an interpretation as a homogenized version of an affine quadric with respect to the unique polar (and thus the center point is unique as well). How can one interpret the general form of a biquadric with a unique center point and learn more about this notion analytically?
- If we still do not want to exclude biquadrics that could serve as biquadrics for *several* possible center points, what is their general form? If we restrict the definition of a biquadric to have one center point, what are more criterions to justify this restriction?
- In order to find flat states for higher prime numbers either analytical work or new calculations on the computer might reveal new insights. But the calculation times in order to test for all possible biquadric fields whether they are flat or not explodes for higher primes numbers then $p = 7$ and hence Monte-Carlo — or other randomized — algorithms might find flat states (or solve otherwise computationally intensive tasks).
- The biquadric field that features a homogenoues distribution of incidences in the whole projective space involves, i.a., biquadrics that could serve for *two* center points. Is it able to find a flat state that only involves biquadrics with a unique center point?
- How can one find at least a small subset of the geometry that can be ordered? There are some ideas, but they have to be made precise!
- It might be possible to find the hyperbolic structure of spacetime in terms of a symmetry-breaking by selecting a hyperplane at infinity. To some extent this would "explain" why there is *time*.
- What are the dynamics of biquadric fields (the analogon to Einstein equations)?
- What are the dynamical equations of matter populating finite projective spaces?

Despite all these open questions this work achieved substantial progress concerning the notion of biquadric fields and supported that biquadric fields might be a reasonable notion.

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Erklärung

Hiermit erkläre ich, dass ich diese Arbeit selbständig angefertigt habe. Ich habe nach bestem Gewissen alle verwendeten Hilfsmittel und Quellen angeben. Des Weiteren wurde diese Arbeit weder einem anderen Prüfungsamt vorgelegt noch veröffentlicht.

Ort, Datum	Unterschrift