Local Domains and Graph Diameter of Biquadric Fields in a Finite Geometry

Master's Thesis in Physics

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Abstract

In modern physics there are two main models describing very accurately different parts of physical phenomena. General relativity uses geometry and the properties of spacetime itself to describe the effects of gravitation where this spacetime is influenced by mass and energy. Quantum mechanics and quantum field theory, on the other hand, describe the physics of elementary and other small particles by the use of a wave function describing the probability amplitude and fields, respectively, of a quantum object. However, these two models describing the same nature, just on different scales, do differ quite a bit in their concepts. The quest of finding a theory that describes both parts, quantum effects and cosmic geometry, is still on.

In this and related work it is tested if such a unification of quantum field theory and general relativity might be based on finite projective geometry [Mecke, 2017]. Therefore, the usual approach of using real (or complex) numbers as number field for coordinates (or wavefunctions) to perform calculations in is left behind in favor of a *Finite Projective Geometry*. The idea is to model spacetime similar to general relativity but based on a finite field such that quantization is not additionally imposed but emerges intrinsically from the finite geometry. Then, singularities and divergences cannot exist neither in a curved spacetime nor in a quantum world modeled over finite fields. However, the quite unusual properties of finite fields require additional care in defining physical quantities. Central to this approach is a 'biquadric' that defines, similar to a metric, an idea of 'closeness'. The long-time goal is to derive the properties of the standard model in a continuum limit for very large finite fields.

In the presented thesis, the *local (world) domain*, defining a subspace in the geometry where an Euclidean-like ordering of the points is possible, and the transformations between different local world domains are investigated.

It is proven that the prior assumption that biquadric points (different from the ones selected as new basis vectors for the local coordinate system) are not part of the local world domain has to be dropped since it is indeed possible that these can be within the local world domain. For the transformation statistics investigating the possible transformations from the local world domain of one local coordinate system into the local world domain of other local coordinate systems some common traits for all considered cases are found. Most of the points within a local world domain can be successfully transformed into other local world domains about two times the prime order p of the Galois field, with a few about twice as much as this. The shape of the distribution resembles a Gaussian distribution and gets narrower for increasing prime numbers p.

Furthermore, the finite field and the neighbouring relations which are defined by the biquadrics are interpreted as a *graph* and its *graph diameter* is explored. It is proven that the diameter of a quadric field is $diam = 2$ in the general case of dimension $d \geq 3$ and $diam \leq 3$ for dimension $d = 2$.

Contents

1 Introduction

In modern physics there are two main models describing extremely accurately different parts of physical phenomena.
General relativity uses geometry and the properties of spacetime itself to describe the effects of gravitation where this spacetime is influenced by mass and energy. It is used to describe effects on a cosmic scale such as the movement of planets and black holes. Quantum mechanics and quantum field theory, on the other hand, describe the physics of elementary and other small particles by the use of wave functions describing the probability amplitude and fields, respectively, of a quantum object. These considered objects, the quantum objects, have both wave and particle properties and some observables in the theory, like the energy, are quantized. Both of these models are very accurate in predicting measurements, but these two models describing the same nature, just on different scales, differ quite a bit in their concepts.
The quest of finding a theory that describes both parts, quantum effects and cosmic geometry, is still on. The currently most prominent theories are loop quantum gravity and string theory.

However, in this and related work a totally different attempt is considered as it is tested if such a unification of quantum field theory and general relativity might be based on finite projective geometry [Mecke, 2017].
The usual approach of using the real (or complex) numbers as the field for coordinates (or wavefunctions) to perform calculations in is left behind in favor of a *Finite Projective Geometry*. The idea is to model spacetime similar to general relativity but based on a finite field such that the quantization leading to the quantum effects of quantum mechanics is not a process done after defining the geometry, but at the beginning. So, the quantization of spacetime is not additionally imposed but emerges intrinsically from the finite projective geometry. Additionally, singularities and divergences (that are a problem in current theories, e.g., when trying to describe the inside of a black hole) cannot exist neither in a curved spacetime nor in a quantum world formulated over finite fields.
These finite fields have properties that are quite different from real fields, e.g., squares can be negative or squaring numbers might not conserve the order of the squared numbers. These different effects require additional care in defining physical quantities, so different mathematical objects and requirements to build a theory able to describe our world were developed. For example, an object called 'biquadric' was defined as a way to have an object that gives, similar to a metric, an idea of 'closeness'.
The long-time goal of the approach is to get to a continuum limit for very large finite fields to compare the behavior of physics in this model with real-world measurements.

In this thesis some considerations regarding these Finite Projective Geometries as a quantum world are presented. In Section 3 the *local (world) domain*, defining a subspace in the geometry where an Euclidean-like ordering of the points is possible, and the transformations between different local world domains are investigated. Furthermore, in Section 4, the finite field and the neighbouring relations which are defined by the biquadrics are interpreted as a *graph* and its diameter is explored.

2 Finite Projective Geometry as a Quantum World

2.1 Finite Fields

First, define the mathematical concepts required for the presented research.

A *field* is a set F which has an addition $+$ and a multiplication $\cdot$ defined on it [Lidl and Niederreiter, 1994]. The addition and the multiplication have the following properties

- both are associative: $a+(b+c)=(a+b)+c$ and $a\cdot(b\cdot c)=(a\cdot b)\cdot c \;\; \forall a,b,c\in F$
- both are commutative: $a+b=b+a$ and $a\cdot b=b\cdot a \;\; \forall a,b\in F$
- identity elements: there is a zero element 0 (neutral element of the addition) such that $a+0=a \;\; \forall a\in F$ and a one element 1 (neutral element of the multiplication) such that $a\cdot 1=a \;\; \forall a\in F$. It is also required that $0\neq 1$.
- inverse elements: there is an inverse element w.r.t. addition $-a$ such that $a+(-a)=0 \;\; \forall a\in F$ and an inverse element w.r.t. multiplication a^{-1} such that $a\cdot a^{-1}=1 \;\; \forall a\in F\backslash\{0\}$
- the multiplication is distributive over the addition: $a\cdot(b+c)=(a\cdot b)+(a\cdot c) \;\; \forall a,b,c\in F$

A *finite field* now is a field that only contains a finite number of elements.

Here, a special kind of finite field is used, namely a *Galois field of order* p, denoted by $\mathbb{F}_p$[1].
This is constructed as follows:
For arbitrary integers a,b and a prime p, say that a is congruent to b modulo p and write $a\equiv b \mod p$, if $a=b+kp$ for some integer k. This congruency is an equivalence relation on the set of integers $\mathbb{Z}$ and partitions $\mathbb{Z}$ into p equivalence classes $[0],[1],[2],...,[p-1]$.
Now, take these equivalence classes $[0],[1],[2],...,[p-1]$ and define the set $\mathbb{Z}/(p)$ by introducing the following binary operations on these equivalence classes:

$$[a]+[b]=[a+b] \tag{1}$$

$$[a]\cdot[b]=[a\cdot b] \quad \forall[a],[b]\in\mathbb{Z}/(p), \quad a,b\in\mathbb{Z}. \tag{2}$$

The set of the equivalence classes $\{[0],[1],[2],...,[p-1]\}$ equipped with the binary relations then forms a field.
Now, for the prime p let $\mathbb{F}_p$ be the set of integers $\{0,1,2,..,p-1\}$ and let $\phi:\mathbb{Z}/(p)\to\mathbb{F}_p$ be a mapping defined by $\phi([a])=a \quad \forall[a]\in\mathbb{Z}/(p), a\in\mathbb{F}_p$.
When equipping $\mathbb{F}_p$ with the field structure inherited from $\mathbb{Z}/(p)$ (which is induced by ϕ) it is also a field, namely the Galois field of order p.

One thing that is particularly different in these prime fields than in the real numbers is the possibility that negative numbers (additive inverse elements) can be square numbers due to the periodicity of the field under multiplication. In particular, also -1 can be a square number. Whether this is realized depends on the prime number p, in fields where $p\equiv 3 \mod 4$ the number -1 is a non-square, but -1 is a square in fields where $p\equiv 1 \mod 4$ [2].
In this work, all fields that will be chosen to perform calculations in will be ones where -1 is a non-square number.

[1] Other notations for the Galois field are $\mathbb{K}_p$ and $\mathbb{Z}_p$.
[2] For example for $p=5$: $-1=\phi([-1])=\phi([-1+5])=\phi([4])=4=2^2$.
The general proof is as follows (restricted to odd primes, i.e., $p>2$):

2.2 A Finite Projective Geometry

A *point-line incidence structure* (incidence geometry) $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is defined as a triple of two sets $\mathcal{P}, \mathcal{L}$, the sets of points and lines, respectively, and an incidence relation $\mathcal{I} \subset \mathcal{P} \times \mathcal{L}$, which is a binary relation indicating which point-line pairs are incident [Moorhouse, 2007]. A point incident with a line is called to be on this line, and this line is called to contain the point.

A *projective plane* is a special point-line incidence structure which has the following properties

- for any two distinct points there is one unique line connecting these points
- for any two distinct lines there exists one unique point which lies on both lines
- there exist at least four points such that no three of them are collinear[3]

An *affine plane* is another special point-line incidence structure. It is different in the second axiom by instead requiring

- for any line L and any point P not on this line L, there is exactly one line L' containing the point P that does not meet the line L

The other two axioms are the same.

A *linear space* is also another special point-line incidence structure which has the following properties

- for any two distinct points there is one unique line connecting these points
- any line contains at least two distinct points

For higher dimensions $d \geq 3$ the modified theorem of Veblen and Young yields another axiom to define a *projective space*

- a linear space is called a projective space if and only if every two-dimensional subspace is a projective plane

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First, define the *Legendre symbol* as

$$\left(\frac{a}{b}\right) = \begin{cases} 1 & \text{if } a \mod b = n^2 \text{ for some integer } n \\ -1 & \text{otherwise} \end{cases} .$$

Then, use the *law of quadric reciprocity* [Gauss, 1986] stating that

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = (-1)^{\frac{a-1}{2}\frac{b-1}{2}}$$

with $a = -1$ and $b = p$ the prime, to calculate $\left(\frac{p}{-1}\right) = \left(\frac{p}{p-1}\right) = 1$, since $p \mod (p-1) = 1 = 1^2$.
Therefore,

$$\left(\frac{-1}{p}\right)\left(\frac{p}{-1}\right) = (-1)^{\frac{-1-1}{2}\frac{p-1}{2}} = (\frac{1}{-1})^{\frac{p-1}{2}} = (-1)^{\frac{p-1}{2}},$$

which is 1, and then $(-1 \mod p)$ a square number, if $\frac{p-1}{2} = 2k$ for some integer k.
This is $p - 1 = 4k \Leftrightarrow p = 1 + 4k \Leftrightarrow p \mod 4 = 1$.
So, if $p \equiv 1 \mod 4$, -1 is a square number. And -1 is not a square number in any other case, which can only be $p \equiv 3 \mod 4$.
Note, that the original proof of the law of quadric reciprocity by Gauss excluded the -1 as a parameter, but this special case was given in an example after the general proof.

[3] on the same line

A *Finite Projective Geometry* is a projective space for which the set of points (and lines) is finite.

There is a large class of projective spaces[4] based on Galois' theorem which will be used in the following work [Mecke, 2017].
To construct these consider the finite Galois field $\mathbb{F}_p$ of order p where p is a prime[5], defined in Section 2.1 above. Use this unique field to define the $(d+1)$-dimensional vector space[6]

$$\mathbb{F}_p^{d+1} = \{(x_0, x_1, ..., x_d) : x_\mu \in \mathbb{F}_p\}. \tag{3}$$

Then, choose the d-dimensional finite projective space $\mathfrak{P}\mathbb{F}^d$ as the point-line incidence structure whose [Richter-Gebert, 2001]

- *points*
  are the 1-dimensional subspaces with the homogeneous coordinates (see below) $\hat{x} = \langle\, x_0, x_1, ..., x_d \,\rangle^t$ with $x_\mu \in \mathbb{F}_p$ (lines in the vector space are points in the projective space)
- *lines*
  are the 2-dimensional subspaces $\{\text{Span}\{\hat{x}^{(1)}, \hat{x}^{(2)}\} : \hat{x}^{(1)}, \hat{x}^{(2)} \in \mathbb{F}_p^{d+1} \backslash \{\hat{0}\}\}$ (planes in the vector space are lines in the projective space)
- *incidence*
  is defined as subspace containment: a point is incident with a line if the underlying 1-dimensional subspaces of the point is contained within the underlying 2-dimensional subspaces of the line

Additionally, one can define *hyperplanes* as the k-dimensional subspaces $\{\text{Span}\{\hat{x}^{(1)}, ..., \hat{x}^{(k+1)}\} : \hat{x}^{(i)} \in \mathbb{F}_p^{d+1} \backslash \{\hat{0}\}\}$.

The *homogeneous coordinates* $\hat{x} = \langle\, x_0, x_1, ..., x_d \,\rangle^t$ with $x_\mu \in \mathbb{F}_p$ are defined as vectors from the vector space $\mathbb{F}_n^{d+1}$ and are a useful way to represent points in the d-dimensional finite projective space $\mathfrak{P}\mathbb{F}^d$ [Richter-Gebert, 2001], so they will be used here.
For these coordinates a point in the projective space $\mathfrak{P}\mathbb{F}^d$ is represented by all vectors that are on the underlying line in $\mathbb{F}_n^{d+1}$.
Since the lines within $\mathbb{F}_n^{d+1}$ that represent the points in the projective space $\mathfrak{P}\mathbb{F}^d$ all go through the origin $\vec{0}$ (they are subspaces of $\mathbb{F}_n^{d+1}$), the whole equivalence class $[v]$ of vectors that only differ by a nonzero multiple λ represent the same point in $\mathfrak{P}\mathbb{F}^d$ as $v = \langle\, v_0, v_1, ..., v_d \,\rangle^t$.

$$[v] = \{v' \in \mathbb{F}_n^{d+1} \ : \ v' = \lambda \cdot v \ \ \forall \lambda \in \mathbb{F} \backslash \{0\}\} \tag{4}$$

When writing a point in $\mathfrak{P}\mathbb{F}^d$ just choose one representative from the equivalence class of vectors $[v]$.

In dimensions $d \geq 3$ every finite projective space is isomorphic to the projective space $\mathfrak{P}\mathbb{F}^d$ over some finite field $\mathbb{F}$.

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[4] It isn't yet proven that all possible projective spaces are known.
[5] In general these Galois fields can be constructed of order n as long as n is the power of a prime $n = p^r$ with $r \in \mathbb{N}$. For reasons of simplicity, only prime numbers $n = p^1$ will be used in the following and no higher powers of primes.
[6] Note that the first component is indexed by a zero.

The point set $\mathcal{P} \equiv \mathfrak{P}\mathbb{F}^d$ of this finite projective space can be decomposed into affine subspaces $\mathbb{F}_p^k$ with $\mathbb{F}_p^0 = \{0\}$ [Mecke, 2018].

$$\mathfrak{P}\mathbb{F}^d = \mathbb{F}_p^d \cup \mathbb{F}_p^{d-1} \cup ... \cup \mathbb{F}_p^1 \cup \mathbb{F}_p^0 \tag{5}$$

An *isomorphism* (*collineation*) of the point-line incidence structure $\phi : (\mathcal{P}, \mathcal{L}, \mathcal{I}) \to (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is defined as a bijective map $\varphi : \mathcal{P} \to \mathcal{P}$ of the set of points for which lines are mapped onto lines. Thus, isomorphic transformations conserve the incidence relations.
These collineations of the projective plane are equivalent to the projective transformations[7] on the projective space [Richter-Gebert and Orendt, 2009].
A *projective transformation* is a transformation that can be written in homogeneous coordinates $\hat{x} = \langle\, x_0, x_1, ..., x_d \,\rangle^t$ with $x_\mu \in \mathbb{F}_p$ by

$$\left\langle \begin{matrix} x_0' \\ ... \\ x_d' \end{matrix} \right\rangle = \begin{pmatrix} \mathbf{A} & \vec{t} \\ \vec{h}^t & c \end{pmatrix} \left\langle \begin{matrix} x_0 \\ ... \\ x_d \end{matrix} \right\rangle \qquad \text{with} \qquad \det \begin{pmatrix} \mathbf{A} & \vec{t} \\ \vec{h}^t & c \end{pmatrix} \neq 0. \tag{6}$$

There are not many properties invariant under projective transformations in the projective space. One important invariant is the *cross ratio*. It is defined for collinear points $\hat{p}_1, \hat{p}_2, \hat{p}_3$ and $\hat{p}_4$ that have line coordinates $z_i \in \mathbb{F}$ (i.e., any order number within a chosen coordinate system) by

$$(\hat{p}_1, \hat{p}_2; \hat{p}_3, \hat{p}_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}. \tag{7}$$

This is the basis for a notion of distance in the projective space.

2.3 Quadrics and Biquadrics

However, we do not have a concept of distance on these objects, yet (only incidence structure). To introduce this concept define objects called ‘quadric’ and ‘biquadric’ that will define neighbouring relations[8] [Mecke, 2017].

A *quadric* Q in $\mathfrak{P}\mathbb{F}^d$ is defined as the set of points for which the homogeneous polynomial $\hat{x}^t\hat{Q}\hat{x}$ of degree 2, that is defined in the $(d+1)$-dimensional homogeneous coordinates $\hat{x} = \langle\, x_0, x_1, ..., x_d \,\rangle^t$ with $x_\mu \in \mathbb{F}_n$, is equal to zero [Beutelspacher and Rosenbaum, 2004] [Moorhouse, 2007].

$$Q = \{\hat{x} : \hat{x}^t\hat{Q}\hat{x} = 0\} \tag{8}$$

with a $(d+1) \times (d+1)$ matrix $\hat{Q} = (Q^{\mu\nu})$, $Q^{\mu\nu} \in \mathbb{F}_p$, called quadric form, that is non-singular and symmetric.

The *dual hyperplane* H to a point $\hat{p}$ is defined as

$$H = \{\hat{x} : \hat{x}^t\hat{Q}\hat{p} = 0\}. \tag{9}$$

[7] The group of all projective transformations is the projective linear group $PGL(d, \mathbb{F})$. It is the linear group of all $d+1$-dimensional invertible matrices $GL(d+1, \mathbb{F})$ up to a scalar multiple, i.e., quotiented by the scalar matrix group $I(d+1, \mathbb{F}) = \{\lambda I_{d+1} \ : \ \lambda \in \mathbb{F}\backslash\{0\}\}$:
$PGL(d, \mathbb{F}) = GL(d+1, \mathbb{F})\backslash I(d+1, \mathbb{F})$ [Mecke, 2018].

[8] The motivation is to define an object that gives, similar to a metric, an idea of ‘closeness’.

A point $\hat{c} \notin Q$ can be chosen as a *center* to the quadric Q. One can choose a center $\hat{c}$ which is symmetric w.r.t. the quadric Q, i.e., there is an involutory perspectivity ('point reflection') with center $\hat{c}$ which leaves the quadric invariant. However, there is no unique choice for such a symmetric center [Laska, 2018].
Since $\hat{c} \notin Q$ and therefore $\hat{c}^t \hat{Q} \hat{c} \neq 0$ one can choose the (irrelevant) normalization of $\hat{Q}$ in such a way that

$$1 = \hat{c}^t \hat{Q} \hat{c} \neq 0. \tag{10}$$

So, when choosing the center as $\hat{c} = \langle\, 0, 0, ..., 0, 1 \,\rangle^t$, the quadric form reads

$$\hat{Q} = \begin{pmatrix} \mathbf{G}_d & \vec{0} \\ \vec{0}^t & -1 \end{pmatrix} \tag{11}$$

with $\mathbf{G}_d$ a $d \times d$-matrix that is non-singular and symmetric.

The *hyperplane at infinity* is defined as the unique hyperplane dual to the center $\hat{c}$

$$H_{\hat{c}}^{\infty} = \{\hat{x} : \hat{x}^t \hat{Q} \hat{c} = 0\}. \tag{12}$$

This plane at infinity $H_{\hat{c}}^{\infty}$ equals the space $\mathbb{F}_p^{d-1} \cup \mathbb{F}_p^{d-2} \cup ... \cup \mathbb{F}_p^1 \cup \mathbb{F}_p^0$ in the decomposition of the projective space in Eq. (5).
In this work, the homogeneous coordinates $\hat{x} = \langle\, x_0, x_1, ..., x_d \,\rangle^t$ with $x_\mu \in \mathbb{F}_p$ are chosen such that the representatives from the equivalence class in Eq. (4) representing points within this hyperplane at infinity $H_{\hat{c}}^{\infty}$ have the last component x_d chosen to $x_d = 0$.

The *affine space* $\mathfrak{A}_{\hat{c}}^d$ with respect to the center $\hat{c}$ now is defined as all points in the projective space $\mathfrak{P}\mathbb{F}^d$ that are not on the hyperplane at infinity $H_{\hat{c}}^{\infty}$.

$$\mathfrak{A}_{\hat{c}}^d = \mathfrak{P}\mathbb{F}^d \backslash H_{\hat{c}}^{\infty} \tag{13}$$

Furthermore, choose the homogeneous coordinates $\hat{x} = \langle\, x_0, x_1, ..., x_d \,\rangle^t$ with $x_\mu \in \mathbb{F}_p$ used here such that points within the affine space are represented by vectors with the last component x_d chosen to $x_d = 1$.

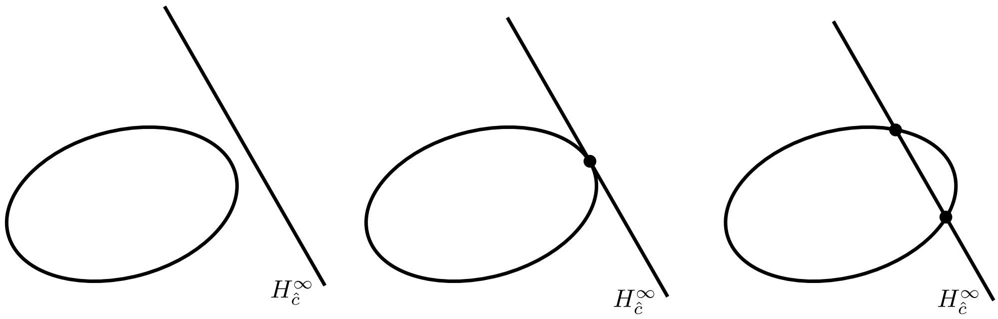
The incidence structure of this affine space is inherited from the structure of the projective space, the sets of points and lines are reduced and lines are truncated. It then is indeed an affine space as defined above in Section 2.1 [Moorhouse, 2007].

This affine space is identified with the d-dimensional vector space $\mathbb{F}_p^d$, if a special coordinate system with center $\hat{c} = \langle\, 0, 0, ..., 0, 1 \,\rangle^t$ is chosen. A point $\hat{p} \in \mathfrak{A}^d$ has homogeneous coordinates

$$\hat{p} = \left\langle \begin{matrix} \mathbf{p}^{(H^\infty)} \\ 1 \end{matrix} \right\rangle \qquad \text{with} \qquad \mathbf{p}^{(H^\infty)} \in \mathbb{F}_p^d. \tag{14}$$

This defines the *inhomogeneous coordinates* (or *affine coordinates*) $\mathbf{p}^{(H^\infty)} \in \mathbb{F}_p^d$ of points $\hat{p}$ in the affine subspace of the projective space $\mathfrak{A}^d \subset \mathfrak{P}\mathbb{F}^d$ with respect to H^∞ [9].

[9] If the current hyperplane at infinity H^∞ is clear from the context the index (H^∞) is dropped and the affine coordinates are just written as $\mathbf{p}$.



(a) elliptic (b) parabolic (c) hyperbolic

Figure 1: Different characteristics of quadrics. These sketches showing quadrics as ellipses meeting the line at infinity (or not) might help to get an intuition for the division in characteristics. The depiction of the quadrics used here does not look like this, for these see Figure 2.

On the affine space one can also define an induced *unit quadric* as the quadric points within the affine space.

$$Q \cap \mathfrak{A}^d = \{\hat{x} \in \mathfrak{A}^d : \hat{x}^t \mathbf{G}_d \hat{x} = 1\} \tag{15}$$

Quadrics can be sorted into three different types or *characteristics*.
One has to be very careful, since the definition of these types, 'elliptic', 'parabolic', and 'hyperbolic', depends on the framework one is in. Both definitions of the characteristics are related but they are *not* the same.
The first possibility of defining the characteristic of a quadric in the affine plane is using the number of points in the whole quadric which lie on the hyperplane at infinity $H_{\hat{c}}^{\infty}$ [Moorhouse, 2007]. There, a quadric with no points at infinity is defined to be 'elliptic', a quadric with one point at infinity is defined as 'parabolic', and a quadric with two points at infinity is defined as 'hyperbolic'[10], see also Figure 1.
The second framework divides the quadrics into the characteristics by the quadric form they are projectively invariant to.
In this framework, depending on the dimension of the space one is in, quadrics can only be of one or two types, respectively [Hirschfeld and Thas, 2016].
In even dimensions there is one possible characteristic and all quadrics are projectively invariant to the form $\hat{P}$, where $\hat{P}$ is

$$\hat{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & ... & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & ... & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & ... & 0 & 0 & 0 \\ & & & & ... & & & \\ 0 & 0 & 0 & 0 & ... & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & ... & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & 2 \end{pmatrix}, \tag{16}$$

[10] It is proven that if three points on a line are within a quadric the whole line is within the quadric [Beutelspacher and Rosenbaum, 2004]. So, a quadric that has three points at infinity is completely within the hyperplane at infinity.

such that the quadratic equation $\hat{x}^t \hat{P} \hat{x} = 0$ is

$$2(x_0 x_1 + x_2 x_3 + ... + x_{d-2} x_{d-1} + x_d^2) = 0. \tag{17}$$

These quadrics are called 'parabolic'.
In odd dimensions there are two invariant types of quadric forms and all quadrics are projectively invariant either to the form $\hat{H}$ or to the form $\hat{E}$, both are defined from

$$\hat{F}_q = \begin{pmatrix} 0 & 1 & 0 & 0 & ... & 0 & 0 \\ 1 & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & 1 & ... & 0 & 0 \\ 0 & 0 & 1 & 0 & ... & 0 & 0 \\ & & & & ... & & \\ 0 & 0 & 0 & 0 & ... & 0 & q \\ 0 & 0 & 0 & 0 & ... & q & 0 \end{pmatrix}, \tag{18}$$

where $\hat{H} = \hat{F}_1$ has $q = 1$ and $\hat{E} = \hat{F}_{\bar{q}}$ has $q = \bar{q}$ with $\bar{q}$ being a non-square in the field. The quadratic equation $\hat{x}^t \hat{F}_q \hat{x} = 0$ then is

$$2(x_0 x_1 + x_2 x_3 + ... + q \cdot x_{d-1} x_d) = 0. \tag{19}$$

One of them is 'hyperbolic' and the other one 'elliptic'.
This form in Eq. (18) can be projectively transformed into the form (the notation here is, for better readability, that all empty matrix entries have value zero)

$$\begin{aligned} \tilde{F}_q &= (T^{-1})^t \ \hat{F}_q \ T^{-1} \\ &= \begin{pmatrix} 1 & & & & ... & & \\ & 1 & & & ... & & \\ & & 1 & & ... & & \\ & & & 1 & ... & & \\ & & & & ... & & \\ & & & & ... & -\frac{1}{2} & -\frac{1}{q} \\ & & & & ... & -\frac{q}{2} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & & & ... & & \\ 1 & 0 & & & ... & & \\ & & 0 & 1 & ... & & \\ & & 1 & 0 & ... & & \\ & & & & ... & & \\ & & & & ... & 0 & q \\ & & & & ... & q & 0 \end{pmatrix} \begin{pmatrix} 1 & & & & ... & & \\ & 1 & & & ... & & \\ & & 1 & & ... & & \\ & & & 1 & ... & & \\ & & & & ... & & \\ & & & & ... & -\frac{1}{2} & -\frac{q}{2} \\ & & & & ... & -\frac{1}{q} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & & & ... & & \\ 1 & 0 & & & ... & & \\ & & 0 & 1 & ... & & \\ & & 1 & 0 & ... & & \\ & & & & ... & & \\ & & & & ... & 1 & 0 \\ & & & & ... & 0 & -q^2 \end{pmatrix}. \end{aligned} \tag{20}$$

This form has the advantage that it preserves a chose affine space if the center $\hat{c} = \langle 0, 0, ..., 0, 1 \rangle^t$ is chosen. This form will be used in Section 4.2.2 later.

The idea of defining such a quadric is to have a set of points that are the next neighbours to the current point (the center).
But these quadrics are porous objects on finite projective spaces: only every second line through the center has a quadric point on it. For the other lines one can go along the whole line without finding a neighbour.
This problem is solved by defining a *biquadric* and taking this object as the one defining neighbouring relations [Mecke, 2017].

A *biquadric* $Q^{\pm}$ with the center $\hat{c} = \langle 0, 0, ..., 0, 1 \rangle^t \in \mathfrak{P}\mathbb{F}^d$ [11] is the set of points in $\mathfrak{P}\mathbb{F}^d$ which fulfill, expressed in homogeneous coordinates, one (or both) of the two equations

$$\hat{x}^t \hat{Q}^+ \hat{x} = 0 \quad \text{and} \quad \hat{x}^t \hat{Q}^- \hat{x} = 0. \tag{21}$$

The quadric forms are

$$\hat{Q}^+ = \begin{pmatrix} \mathbf{G}_d & \vec{0} \\ \vec{0}^t & f_+^2 \end{pmatrix} \quad \text{and} \quad \hat{Q}_0^- = \begin{pmatrix} \mathbf{G}_d & \vec{0} \\ \vec{0}^t & -f_-^2 \end{pmatrix}, \tag{22}$$

with $\mathbf{G}_d$ a $d \times d$-matrix that is non-singular and symmetric, and squares $f_\pm^2 \in \mathbb{F}_p$.
So,

$$Q^{\pm} = \{\hat{x} : \hat{x}^t \hat{Q}^+ \hat{x} = 0 \quad \text{and / or} \quad \hat{x}^t \hat{Q}^- \hat{x} = 0\}. \tag{23}$$

All biquadric points $\hat{x} = \langle \mathbf{x}, 1 \rangle$ fulfill one of the two relations in inhomogeneous coordinates $\mathbf{x}$

$$f_\pm^2 = \mp \mathbf{x}^t \mathbf{G}_d \mathbf{x}. \tag{24}$$

The quadric forms $\hat{Q}^{\pm}$ can always be scaled in such a way that $f_+ = 1$.
Then, the *effective scaling factor* $f = -f_-^2$ has to be a non-square in the field[12].
Using this, the biquadric has the following form

$$\hat{Q}^+ = \begin{pmatrix} \mathbf{G}_d & \vec{0} \\ \vec{0}^t & 1 \end{pmatrix} \quad \text{and} \quad \hat{Q}^- = \begin{pmatrix} \mathbf{G}_d & \vec{0} \\ \vec{0}^t & f \end{pmatrix}. \tag{25}$$

The matrix $\mathbf{G}_d$ induces a metric within the affine space defining the distance between two affine points via the cross ratio defined in Eq. (7).

It is also possible for biquadric points to be on the hyperplane at infinity. In this case, the line connecting the center and this biquadric point at infinity is defined as the *light cone* L_c. (Traveling to (time) distance one in the light cone along the direction of the light cone should take you to infinity when seen from the outside.)

For a biquadric, other than for a quadric, *each* line going through the center intersects the biquadric exactly twice, in both directions from the center ('forwards' and 'backwards' on the line).
This biquadric structure now gives a notion of distance in the finite projective geometry: all biquadric points $\hat{q}$ are defined as nearest neighbors to the center $\hat{c}$ and their distance s to the center is defined as $s = 1$.
However, their affine coordinates $\mathbf{q}$ can be arbitrarily large (as long as within the field) and arbitrarily ordered.

A *biquadric field* is such that for every point one has a biquadric that has this point as a center $\hat{c}$ and then defines neighbours for this point.

[11] A general biquadric centered at any point other than the special center $\hat{c} = \langle 0, 0, ..., 0, 1 \rangle^t$ can be obtained by a translation of this biquadric (this is shown, e.g., for quadrics in Section 4.2.1). Usually, a biquadric with center $\hat{p}$ is noted as $Q_{\hat{p}}^{\pm}$ and for the special center $\hat{c} = \langle 0, 0, ..., 0, 1 \rangle^t$ as $Q_0^{\pm}$. Here, drop the index 0 for $Q_0^{\pm}$ for simplicity and mark the biquadric if it has a different center.
[12] We are restricting to fields where -1 is a non-square. There -1 times a square number is a non-square.

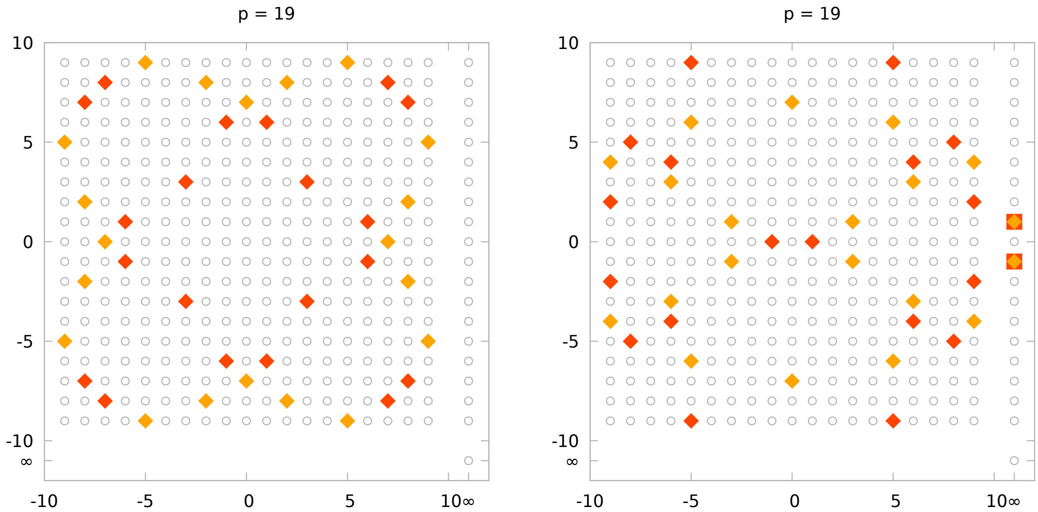


Figure 2: *left*: Plot of the biquadric $Q_1^\pm$ as in Eq. (26). *right*: Plot of the biquadric $Q_2^\pm$ as in Eq. (27).
The red points are the ones in Q^+, the orange ones are in Q^-. The points not on the hyperplane (here just a line) at infinity are within the affine space.
On the left side both quadrics are elliptic w.r.t. the characteristic criterion defined by number of points on the line at infinity. On the right side, both are hyperbolic.

Figure 2 shows the plot of two biquadrics $Q_1^\pm$ and $Q_2^\pm$ with the quadric forms shown in Eqs. (26) and (27) in dimension $d = 2$ for the projective space with prime number $p = 19$.

$$\hat{Q}_1^+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \hat{Q}_1^- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{pmatrix} \tag{26}$$

$$\hat{Q}_2^+ = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \hat{Q}_2^- = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{pmatrix} \tag{27}$$

3 The Local World Domain

Finite geometries do have the property that squaring numbers does not preserve the order of these numbers. Because of the periodic boundary the square of a initially larger number can be smaller than the square of another, initially smaller number.
But when doing the considerations only on a restricted part of the geometry this unwanted property can be avoided by selecting only all the points with coordinates small enough that taking the square of numbers preserves their order. This restricted domain is called *local (world) domain* [Mecke, 2017].

3.1 Definition of the Local World Domain

To obtain a *local world domain*, first choose a new *local coordinate system* where the coordinate distances between the center and biquadric points in the direction of the coordinate axes is equal to the defined physical one, i.e., $|\mathbf{q} - \mathbf{c}| = 1$ [13].
Then, restrict all considerations to a smaller set of points which all in this coordinate system have coordinates *close* to the center (see below for the definition of 'close'). Within this smaller set the order of points is conserved under taking the square. Therefore the points are well-ordered in a Euclidean sense. This smaller set of points then is the *local world domain*.

The construction process is as follows [Mecke, 2018]:
The starting coordinate system is called *standard basis*. In this system the components of the biquadrics are given.
In a d-dimensional space then choose the new coordinate system by first selecting d different directions.
These correspond to d different lines L and their corresponding unit points $\pm\hat{q}_L \in Q^{\pm}$. The unit point on the line L is the one biquadric point that can be found when going from the center in the direction of the line L. Since we have a biquadric there is exactly one per sign, i.e., direction on the line. Per definition, these unit points have distance 1.
This choice of d different directions is restricted by the requirement that the chosen biquadric points have to be within the affine space, i.e., the chosen directions must not be along the light cone L_c. Furthermore, choose them such that one biquadric point is from the one quadric Q^+ and all $d-1$ other points are from the other one Q^-, or choose them the other way round such that one is from Q^- and all others from Q^+.
These chosen different directions now become the new axis lines of the new coordinate system[14].
Therefore, first number the new axes by enumerating the chosen biquadric points by $\pm\hat{q}^\alpha$ with $\alpha = 0, ..., d$ [15] and then normalizing them as $\hat{e}^\alpha = \hat{c} + (\hat{q}^\alpha - \hat{c})/f$, with f being the effective scaling factor (the non-square) that occurs in the definition of the biquadric forms in Eq. (25)[16].
This local coordinate system spanned by the normalized unit points allows a local ordering of the points $\hat{p} \in \mathfrak{P}\mathbb{F}^d$ in a restricted smaller set of points.
This set of *close* points is defined as all points $\hat{p} = \langle\, p_0, p_1, ..., p_{d-1}, 1\,\rangle$ whose affine coordinates fulfill $p_\nu^2 \leq \frac{p-1}{2}\ \forall \nu \in \{0, 1, ..., d-1\}$, with p the prime number of the finite field $\mathbb{F}_p$. In this equation, the product p_ν^2 is calculated on $\mathbb{Z}$.

[13] The physical idea is to have space quantized to the Planck length ℓ_P which is set to 1 here [Laska, 2018].

[14] Note that when choosing $\hat{q}^a$ and $\hat{q}^b$ for a first local coordinate system, and then $\hat{q}^a$ and $-\hat{q}^b$ (or $-\hat{q}^a$ and $\hat{q}^b$) for another one, this local coordinate system is the same just reflected at the axis. When choosing $-\hat{q}^a$ and $-\hat{q}^b$ for the second one, it is the point reflection (through the center) of the first one.

[15] When denoting the *number* of a basis vector the index α will be used. For the *components* of a vector the index ν will be used.

[16] This procedure describes the normalization of the basis vectors from *both* quadrics. Below in Section 3.4.3 also other possibilities are discussed.

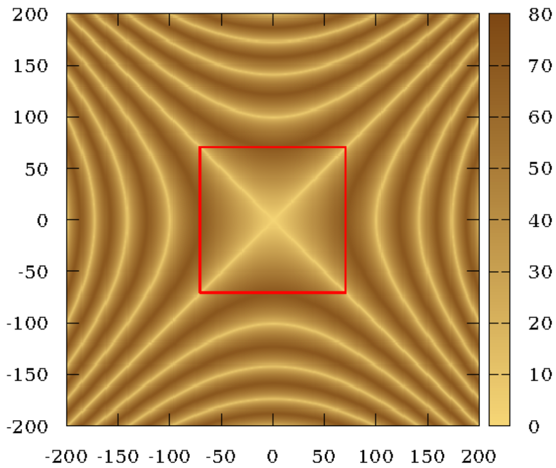


Figure 3: Inside the red box one can see the local world domain for $p = 10007$ for a hyperbolic biquadric [here 'hyperbolic' is w.r.t. to the first framework that is introduced in Section 2.3. This definition can not only be made for quadrics but also for biquadrics, where a 'hyperbolic' biquadric is defined as one where both underlying quadrics have the same two points at infinity.]. The range of the plot is restricted to show the local world domain better. Inside this domain the distances s of a point $\mathbf{p}$ to the center $\mathbf{c}$ $\left(s(\mathbf{p}) = \sqrt[\mathbb{R}]{|\mathbf{G}_{2\,ij}p^i p^j \mod p|}\right)$ which are encoded in colors behave as we know it from our real world. Outside the local domain the periodicity of the prime field destroys the order of distances.
Picture and description (slighly changed to fit the notation here) taken from [Gimperlein, 2018].

As long as these products of coordinate components are smaller than $\frac{p-1}{2}$, the points are well ordered in the Euclidean sense with the metric induced by the biquadric (see also Figure 3). So, within this set the biquadric naturally induces a finite Euclidean-like geometry.
This local region around the center $\hat{c}$ is called *local world domain* $\mathcal{D}_{\hat{c}}$.

$$\mathcal{D}_{\hat{c}} = \{\hat{p} = \left\langle \begin{matrix} \mathbf{p} \\ 1 \end{matrix} \right\rangle \;:\; |p_\nu| \leq \sqrt{\frac{p-1}{2}} \quad \forall \nu \in \{0, 1, ..., d-1\}\} \tag{28}$$

The assumption made *prior to this work* was that except the unit points $\pm\hat{q}^\alpha$ all other biquadric points $\hat{q} \in Q^\pm$ do not belong to this local world domain (although their distance to the center is 1 as for $\pm\hat{q}^\alpha$). This would be formulated as the condition shown in Eq. (29).

$$\hat{p} \in \mathcal{D}_{\hat{c}}^{\text{prior}} \quad \Rightarrow \quad \hat{p} \notin Q^\pm \backslash \{\pm\hat{q}^\alpha\} \tag{29}$$

This work, however, will show that there are some other quadric points transformed into the local world domain. Therefore, the assumption of quadric points that are not chose as new basis vectors not being part of the local world domain has to be dropped.

3.2 The Coordinate Transformations (for $d = 2$)

First, introduce the notation for coordinate transformations of points from the standard basis into a local coordinate system and between different local coordinate systems.
The standard basis may be written in un-primed coordinates $\mathbf{p}$ and the new local coordinate system (built from the quadric points as new basis vectors) in primed coordinates $\mathbf{p}'$.
The general equation for an *affine* linear coordinate transformation is

$$\mathbf{p}' = \Theta\mathbf{p} + \mathbf{B} \tag{30}$$

with Θ an invertible matrix and $\mathbf{B}$ a translation vector.
Here, $\mathbf{B} = \vec{0}$ since the center is conserved.
Furthermore, all considerations are restricted to the 2-dimensional case, so the considered transformations are in the affine space of a 2-dimensional projective space.
Then,

$$\mathbf{p}' = \Theta\ \mathbf{p} \tag{31}$$

and backwards

$$\mathbf{p} = \Theta^{-1}\ \mathbf{p}'. \tag{32}$$

We know that the newly chosen basis vectors (which are the biquadric points in the chosen directions), denoted as $\mathbf{e}^0$ and $\mathbf{e}^1$ in the standard basis, do have the components $\mathbf{e}'^0 = (1,0)^t$ and $\mathbf{e}'^1 = (0,1)^t$ in the new basis. This helps to find the matrix Θ^{-1}.

$$\mathbf{e}^0 = \begin{pmatrix} e^0_0 \\ e^0_1 \end{pmatrix} = \Theta^{-1}\mathbf{e}'^0 = \begin{pmatrix} (\Theta^{-1})_{00} & (\Theta^{-1})_{01} \\ (\Theta^{-1})_{10} & (\Theta^{-1})_{11} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} (\Theta^{-1})_{00} \\ (\Theta^{-1})_{10} \end{pmatrix}$$

$$\mathbf{e}^1 = \begin{pmatrix} e^1_0 \\ e^1_1 \end{pmatrix} = \Theta^{-1}\mathbf{e}'^1 = \begin{pmatrix} (\Theta^{-1})_{00} & (\Theta^{-1})_{01} \\ (\Theta^{-1})_{10} & (\Theta^{-1})_{11} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} (\Theta^{-1})_{01} \\ (\Theta^{-1})_{11} \end{pmatrix}$$

$$\Rightarrow \Theta^{-1} = \begin{pmatrix} e^0_0 & e^1_0 \\ e^0_1 & e^1_1 \end{pmatrix} \tag{33}$$

$$\Rightarrow \mathbf{p} = \begin{pmatrix} e^0_0 & e^1_0 \\ e^0_1 & e^1_1 \end{pmatrix} \mathbf{p}' \tag{34}$$

And calculating the inverse of a 2×2 matrix gives the backwards transformation.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\Rightarrow \mathbf{p}' = \Theta\ \mathbf{p} = \frac{1}{e^0_0 e^1_1 - e^1_0 e^0_1} \begin{pmatrix} e^1_1 & -e^1_0 \\ -e^0_1 & e^0_0 \end{pmatrix} \mathbf{p} \tag{35}$$

The transformation of a point from one local coordinate system into another is composed of a transformation from the first local coordinate system (notated with one prime) into the standard basis and then into the other local coordinate system (denoted with two primes).
The two transformations differ in the matrix Θ whose components are the coordinates of the new basis vectors that were chosen for the different local coordinate systems.
Write the matrix to transform into the first system as Θ' and the one to transform in the second one as Θ''.

Then, one has

$$\begin{aligned}\mathbf{p}'' &= \Theta''\ \mathbf{p} \\ &= \Theta''\ (\Theta')^{-1}\ \mathbf{p}' \qquad (36)\\ &= \frac{1}{e''^0_0 e''^1_1 - e''^1_0 e''^0_1} \begin{pmatrix} e''^1_1 & -e''^1_0 \\ -e''^0_1 & e''^0_0 \end{pmatrix} \begin{pmatrix} e'^0_0 & e'^1_0 \\ e'^0_1 & e'^1_1 \end{pmatrix} \mathbf{p} \qquad (37)\end{aligned}$$

3.3 Example Local World Domain in the Standard Basis (for $d = 2$)

Now, an example of a local world domain in a local coordinate system and its representation in the standard basis is introduced.
For this example, the dimension is $d = 2$ and the prime number is chosen to $p = 23$. The center is chosen as $\hat{c} = \langle\, 0, 0, 1\, \rangle$ and the biquadric $Q^{\pm}$ has the quadric forms, represented in the standard basis,

$$\hat{Q}^{+} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \hat{Q}^{-} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{pmatrix}, \qquad (38)$$

where $\hat{Q}^{-}_{22} = f = 10$ [17] is a non-square in $\mathbb{F}_{23}$. All considerations are restricted to the affine space since the local world domain is just within the affine.
Now, the following steps are taken to calculate the components of the points in the local world domain of an arbitrarily selected local coordinate system in the standard basis:

- ◇ the coordinates of all affine points in the quadrics Q^{+} and Q^{-} are calculated (in the standard basis) by checking the condition $\hat{p} \in Q^{\pm} \;\Leftrightarrow\; \hat{p}^t \hat{Q}^{\pm} \hat{p} = 0$ for every point.
- ◇ two quadric points (because we have $d = 2$) are selected as new basis vectors. The choice is arbitrary, only restricted by the requirement that the points have to be from the affine space and one point is from Q^{+} and the other one from Q^{-}.

  The (arbitrarily) chosen points are $\hat{q}^0 = \left\langle \begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix} \right\rangle \in Q^{+} \cap \mathfrak{A}^2$ and $\hat{q}^1 = \left\langle \begin{smallmatrix} 0 \\ 6 \\ 1 \end{smallmatrix} \right\rangle \in Q^{-} \cap \mathfrak{A}^2$.
- ◇ the normalized unit points for the quadric points are calculated. This is calculating $\hat{e}^{\alpha}$ as $\hat{e}^{\alpha} = \hat{c} + (\hat{q}^{\alpha} - \hat{c})/f$, or in affine components $\mathbf{e}^{\alpha} = \mathbf{q}^{\alpha}/f$, since the center is $\hat{c} = \langle\, 0, 0, 1\, \rangle$. So, the new basis vectors are

$$\hat{e}^0 = \left\langle \begin{smallmatrix} 7 \\ 0 \\ 1 \end{smallmatrix} \right\rangle \quad \text{and} \quad \hat{e}^1 = \left\langle \begin{smallmatrix} 0 \\ -4 \\ 1 \end{smallmatrix} \right\rangle. \qquad (39)$$

- ◇ a generic local world domain set is defined as all affine coordinates $\mathbf{g}$ which fulfill the relation $g_{\nu}^2 \leq \frac{p-1}{2}\ \forall \nu \in \{0, 1\}$. This set here is just a set of *coordinate components* that fulfill this restriction. It does not represent any points, but is just a set made to compare with the coordinates of actual points to see whether they are in the local world domain of the local coordinate system they are currently represented in.
- ◇ the coordinates of the points in the local world domain of the selected local coordinate system are calculated in the standard basis. For this, first go into the local coordinate

[17] Remember, the first components are indexed by 0.

system. There all the points that are within the local world domain do have the components that are also saved in the generic local world domain set. So this set of components is transformed from the local coordinate system into the standard basis using Θ^{-1}.

The standard basis components of the two new basis vectors that were selected before define the components of Θ^{-1}. Doing the transformation calculations for the affine components is sufficient since the affine space is conserved.

In the following calculations write components of points in the standard basis without a prime and components in the local coordinate system with a prime.

$$\hat{p} = \begin{pmatrix} p_0 \\ p_1 \\ 1 \end{pmatrix}_{\text{standard basis}} = \begin{pmatrix} p'_0 \\ p'_1 \\ 1 \end{pmatrix}_{\text{basis of local coordinate system}}$$

So, for each point $\hat{p}$ one has

affine components of $\hat{p}$ in the standard basis:	p_0 and p_1
affine components of $\hat{p}$ in the new basis of the local coordinate system:	p'_0 and p'_1

Then, the transformation from the local coordinate system into the standard basis is (see Section 3.2)

$$\begin{aligned} \mathbf{p} = \Theta^{-1}\ \mathbf{p}' &= \begin{pmatrix} e_0^0 & e_0^1 \\ e_1^0 & e_1^1 \end{pmatrix}\ \mathbf{p}'. \\ \Rightarrow p_0 &= p'_0 e_0^0 + p'_1 e_0^1 \\ p_1 &= p'_0 e_1^0 + p'_1 e_1^1 \end{aligned} \tag{40}$$

◇ now, both the quadric points of the two quadrics and all points that are within this local world domain of the selected local coordinate system are presented in the same basis (the standard basis) and plotted.

Figure 4 shows the quadric points and all points that are within the local world domain of this selected local coordinate system, both represented in the standard basis.
Important to notice are the points $\left\langle \begin{matrix} \pm 7 \\ \pm 4 \\ 1 \end{matrix} \right\rangle$ and $\left\langle \begin{matrix} \mp 7 \\ \pm 4 \\ 1 \end{matrix} \right\rangle$. Since both sets are represented in the same basis, if two points are shown at the same position they are the same point. So these points are both a quadric point and within the local world domain.
An analytic calculation for the point $\hat{p} = \left\langle \begin{matrix} 7 \\ 4 \\ 1 \end{matrix} \right\rangle$ shows that it is indeed both, a quadric point and transformed into the local world domain.

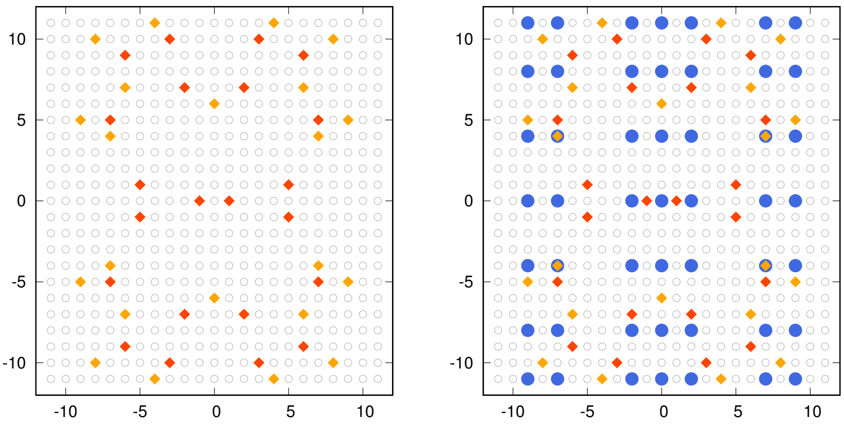


Figure 4: *left:* Biquadric points to the biquadric with the quadric forms in Eq. (38). *right:* Same biquadric points and the local world domain.
The red points are the ones in Q^+, the orange ones are in Q^-. The blue points are the ones that are transformed into the local world domain for the choice of new basis vectors shown in Eq. (39).

Test for being a quadric point:

$$
\begin{aligned}
\hat{p} \in Q^- \;\Leftrightarrow\; & \hat{p}^t \hat{Q}^- \hat{p} = 0 \\
\hat{p}^t \hat{Q}^- \hat{p} = & \left\langle \begin{matrix} 7 \\ 4 \\ 1 \end{matrix} \right\rangle^t \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{pmatrix} \left\langle \begin{matrix} 7 \\ 4 \\ 1 \end{matrix} \right\rangle = \left\langle \begin{matrix} 7 \\ 4 \\ 1 \end{matrix} \right\rangle^t \cdot \left\langle \begin{matrix} -7 \\ 4 \\ 10 \end{matrix} \right\rangle \\
& = -49 + 16 + 10 \\
& = 20 + 16 + 10 \\
& = 46 \\
& = 0,
\end{aligned}
\tag{41}
$$

since $-49 \mod 23 = 20$ and $46 \mod 23 = 0$.

Test (in affine components) for being in the local world domain when transformed into the selected local coordinate system:

$$
\begin{aligned}
\hat{e}^0 &= \left\langle \begin{matrix} 7 \\ 0 \\ 1 \end{matrix} \right\rangle,\ \hat{e}^1 = \left\langle \begin{matrix} 0 \\ -4 \\ 1 \end{matrix} \right\rangle \text{ and } \hat{p} = \left\langle \begin{matrix} 7 \\ 4 \\ 1 \end{matrix} \right\rangle \\
\mathbf{p}' &= \Theta\ \mathbf{p} \\
&= \frac{1}{e_0^0 e_1^1 - e_0^1 e_1^0} \begin{pmatrix} e_1^1 & -e_0^1 \\ -e_1^0 & e_0^0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \\
&= \frac{1}{e_0^0 e_1^1 - e_1^0 e_0^1} \begin{pmatrix} p_0 e_1^1 - p_1 e_0^1 \\ -p_0 e_1^0 + p_1 e_0^0 \end{pmatrix} \\
&= \frac{1}{7 \cdot (-4) - 0} \begin{pmatrix} 7 \cdot (-4) - 0 \\ -0 + 4 \cdot 7 \end{pmatrix} \\
&= \frac{1}{133} \begin{pmatrix} 133 \\ 28 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&\Rightarrow \quad {p'_0}^2 = 1 \leq 11 = \frac{23-1}{2} = \frac{p-1}{2} \\
&\text{and} \quad {p'_1}^2 = 1 \leq 11, \qquad (42)
\end{aligned}
$$

since $\frac{28}{133} \mod 23 = \frac{5}{18} \mod 23 = -1$.

Therefore, the assumption that quadric points (different from the ones chosen as new basis vectors) are not part of the local world domain has to be dropped.

3.4 Transformation Statistics - Calculating How Often Any Point that is Part of a Local World is Transformed into Another Local World Domain (for $d = 2$)

Now, it is calculated how often any point that is part of the local world domain in one local coordinate system is successfully transformed into the local world domain of another local coordinate system.
Therefore, for a prime p and a chosen biquadric $Q^{\pm}$, one local coordinate system is chosen (arbitrarily) as an example system to do the considerations in. It will be called 'selected coordinate system', and its local world domain 'selected local world domain'.
Next, all other possible ways of constructing a local coordinate system are executed by calculating all possible combinations of d (here $d = 2$) new basis vectors from the biquadric points. Then, it is counted how often any point $\hat{p}$ that is in the local world domain of one of these local coordinate systems is transformed into the local world domain of the selected local coordinate system. This number of possible transformations is depicted as the value in a *heatmap* representing the points in the selected local world domain.

3.4.1 Calculation of the Heatmaps

The heatmap H represents the local world domain of the selected local coordinate system. The value of each point in the heatmap shows how often this point in the selected local world domain can be successfully transformed into another local world domain of another local coordinate system.
Affine coordinates in the selected local coordinate system may be written with one prime as $\mathbf{p'}$, affine coordinates in another local coordinate system, called 'current local coordinate system', may be written with two primes as $\mathbf{p''}$.
Each field H_{ij} of the heatmap H represents the point

$$\hat{p} = \begin{pmatrix} i \\ j \\ 1 \end{pmatrix}_{\text{basis of selected local coordinate system}} \qquad \text{with } \mathbf{p'} = (i, j)^t.$$

The value of H_{ij} is calculated by a sum over all possible other local coordinate systems and their local domains, with each term giving a contribution of 1 if the current point in the local world domain of the current local coordinate system is transformed onto the point $\mathbf{p'} = (i, j)^t$, and no contribution if not.
So,

$$H_{ij} = \sum_{(\mathtt{I},\mathtt{J})\in\mathtt{C}} \sum_{\mathbf{p''}\in G} \delta_{i\ \ p'^{0}(\mathbf{p''},I,J)} \quad \delta_{j\ \ p'^{1}(\mathbf{p''},I,J)} \tag{43}$$

where

- I is the index of the new basis vector from $Q^+ \cap \mathfrak{A}^2$ for the *current* local coordinate system
- J is the index of the new basis vector from $Q^- \cap \mathfrak{A}^2$ for the *current* local coordinate system
- I_s / J_s is the index of the new basis vector from $Q^+ \cap \mathfrak{A}^2$ / $Q^- \cap \mathfrak{A}^2$ for the *selected* local coordinate system
- $\mathtt{C} = \{(I, J)\ : I \in \{1, 2, ..., |Q^+ \cap \mathfrak{A}^2|\} \setminus I_s$ and $J \in \{1, 2, ..., |Q^- \cap \mathfrak{A}^2|\} \setminus J_s\ \}$ is the set of all possible allowed combinations for new basis vectors for current local coordinate systems (I_s and J_s are excluded)

- $G = \{\mathbf{g} \ : |g_\nu| \leq \sqrt{\frac{p-1}{2}} \quad \forall \nu \in \{0,1\}\}$ is the set of components that fulfill the local world domain condition. (In any local coordinate system the points in the local world domain do have exactly these components.)
- $\mathbf{p}'(\mathbf{p}'', I, J) = (p'^0(\mathbf{p}'', I, J),\ p'^1(\mathbf{p}'', I, J))^t = \Theta^{-1}_{I_s J_s} \Theta_{IJ}\ \mathbf{p}''$ are the components in the selected local coordinate system of the point that has components $\mathbf{p}''$ in the current local coordinate system that is determined by the new basis vectors indexed by I and J (the Ith point from $Q^+ \cap \mathfrak{A}^2$ and the Jth point from $Q^- \cap \mathfrak{A}^2$)
- Θ_{IJ} is the transformation matrix for the transformation from the current local coordinate system that is determined by the basis vectors indexed by I and J to the standard basis

$$\Theta_{IJ} = \begin{pmatrix} e_0^I & e_0^J \\ e_1^I & e_1^J \end{pmatrix} \tag{44}$$

- $\Theta^{-1}_{I_s J_s}$ is the transformation matrix for the transformation from the standard basis into the selected local coordinate system

$$\Theta^{-1}_{I_s J_s} = \frac{1}{e_0^{I_s} e_1^{J_s} - e_0^{J_s} e_1^{I_s}} \begin{pmatrix} e_1^{J_s} & -e_0^{J_s} \\ -e_1^{I_s} & e_0^{I_s} \end{pmatrix} \tag{45}$$

So,

$$\mathbf{p}'(\mathbf{p}'', I, J) = \Theta^{-1}_{I_s J_s} \Theta_{IJ}\ \mathbf{p}'' = \frac{1}{e_0^{I_s} e_1^{J_s} - e_0^{J_s} e_1^{I_s}} \begin{pmatrix} e_1^{J_s} & -e_0^{J_s} \\ -e_1^{I_s} & e_0^{I_s} \end{pmatrix} \begin{pmatrix} e_0^I & e_0^J \\ e_1^I & e_1^J \end{pmatrix} \mathbf{p}''. \tag{46}$$

Then, H_{ij} is the sum of all current local coordinate systems indexed by $(I, J) \in \mathtt{C}$ (that are different from the selected one) over the question if one of the points in the current local world domain indexed by (I, J) is transformed onto the exact point $(i, j)^t$ in the selected local world domain indexed by (I_s, J_s).
The sum

$$\sum_{\mathbf{p}'' \in G} \delta_{i\ p'^0(\mathbf{p}'', I, J)} \quad \delta_{j\ p'^1(\mathbf{p}'', I, J)} \tag{47}$$

is at most one, since the transformation is bijective, so at most one of the $\mathbf{p}'' \in G$ is transformed onto $(i, j)^t$.

3.4.2 Description of the Procedure

Step-by-step description of the procedure calculating the heatmaps:

- the dimension is set to $d = 2$ ($\mathfrak{P}\mathbb{F}^2$ and $\mathfrak{A}^d = \mathbb{F}^2$)
- the center is chosen as $\hat{c} = \langle\, 0, 0, 1\, \rangle$
- the free parameters for each calculation are the prime number p and the free components of the two quadric forms $\hat{Q}^+$ and $\hat{Q}^-$ building the biquadric $Q^\pm$. The free components are the components of the matrix $\mathbf{G}_2$ that is defined in Eq. (25) and the effective scaling factor f which has to be a non-square scalar from $\mathbb{F}_p$. The quadric forms are of the form

$$\hat{Q}^+ = \begin{pmatrix} \hat{Q}_{00} & \hat{Q}_{01} & 0 \\ \hat{Q}_{01} & \hat{Q}_{11} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \hat{Q}^- = \begin{pmatrix} \hat{Q}_{00} & \hat{Q}_{01} & 0 \\ \hat{Q}_{01} & \hat{Q}_{11} & 0 \\ 0 & 0 & f \end{pmatrix}. \tag{48}$$

- ◇ the coordinates of all affine points in the quadrics Q^+ and Q^- are calculated (in the standard basis) by checking the condition $\hat{p} \in Q^{\pm} \;\Leftrightarrow\; \hat{p}^t \hat{Q}^{\pm} \hat{p} = 0$ for every point

- ◇ the normalized unit points associated to the quadric points are calculated for each quadric point. This is calculating $\hat{e}^{\alpha} = \hat{c} + (\hat{q}^{\alpha} - \hat{c})/f$, or in affine components $\mathbf{e}^{\alpha} = \mathbf{q}^{\alpha}/f$, since the center is $\hat{c} = \langle\, 0, 0, 1\,\rangle$. Later, α will be $\in \{0, 1\}$, but here just all possible normalized unit points are calculated.

  Normalizing the new basis vectors from both quadrics is not the only choice one can make. The two other valid normalization schemes are 1) to normalize only the basis vectors generated from Q^- and 2) to not normalize any basis vector. This is discussed in detail in Section 3.4.3 below.

- ◇ one pair of these normalized quadric points is selected and ordered to become the basis for the selected local coordinate system the local world domain is viewed in. The choice is arbitrary, only restricted by the requirement that the points have to be from the affine space and one point is from Q^+ and the other one from Q^-.

- ◇ a generic local world domain set is defined as all coordinates $\mathbf{g}$ which fulfill the relation $g_{\nu}^2 \leq \frac{p-1}{2} \; \forall \nu \in \{0, 1\}$. This set here is again just a set of *coordinate components* that fulfill this restriction.

- ◇ the heatmap H showing how often any point in the local world domain of the selected local coordinate system is transformed into another local world domain (of another local coordinate system) is calculated as in Eq. (43). Therefore, each point from the local world domain of any other local coordinate system is transformed into the standard basis and then into the selected one. Not all transformed points will fulfill the requirement ${p'}_{\nu}^2 \leq \frac{p-1}{2} \; \forall \nu \in \{0, 1\}$ with their coordinates in the basis of the selected local coordinate system, i.e., not all of them will be part of the selected local world domain. The ones that are part of it, lead to an increase of the heatmap value at their position.

  So, at the end, a value of the heatmap shows, how often points from other local world domains are transformed onto its position, or equivalently, how often this point is transformed into another local world domain.

- ◇ the just calculated heatmap has to be processed a bit, since the heatmap values at H_{0j} and H_{i0} are also increased in the case when the current local coordinate system and the selected one have one basis vector $\hat{e}^{I_s} = \hat{e}^{I} = \hat{e}_c$ in common (this is allowed as long as the second basis vector is different $\hat{e}^{J_s} \neq \hat{e}^{J}$, or, when the other way round $\hat{e}^{J_s} = \hat{e}^{J} = \hat{e}_c$, $\hat{e}^{I_s} \neq \hat{e}^{I}$). In this case all points that are, when written as linear combination of the basis vectors of the current local world domain of the form $\hat{p} = \lambda \cdot \hat{e}_c + 0 \cdot \hat{e}^{J_s}$, are trivially transformed into the selected local world domain. So, these cases are chosen to be removed from the heatmap.

  There they are of the form $\mathbf{p}' = (p_0, 0)^t$ or $\mathbf{p}' = (0, p_1)^t$, i.e., they are positioned on the coordinate axes. For a point of the form $(p_0, 0)^t$ this occurs as often as there are basis vectors from Q^-, minus 1 (the one is the basis vector for the selected local world domain, it is already excluded before). For a point of the form $(0, p_1)^t$ it occurs as often as there are basis vectors from Q^+, minus 1. So the heatmap values of axis points are decreased by $|Q^{\pm} \cap \mathfrak{A}^2| - 1$ to handle these artifacts.

- ◇ the heatmap value of the center $\hat{c} = \langle\, 0, 0, 1\,\rangle$ is artificially set to zero, since it occurs in every local world domain

Therefore, any value in the heatmap shows how many transformations into other local coordinate systems (with basis vectors created from quadric points) can be found, such that this point is also part of the local domain in the new coordinate system.

3.4.3 Normalization of the new Basis Vectors

The procedure described above requires a normalization of the basis vectors from both quadrics with the effective scaling factor f.
In the calculations used here, a normalization of the basis vectors from both quadrics is equivalent to not normalizing at all, since $\Theta^{-1}_{I_sJ_s}\Theta_{IJ}$ is invariant under a common normalization of all new basis vectors, and therefore the whole heatmap is invariant.

$$\begin{aligned}
\Theta^{-1}_{I_sJ_s}\Theta_{IJ} &= \frac{1}{e_0^{I_s}e_1^{J_s} - e_0^{J_s}e_1^{I_s}} \begin{pmatrix} e_1^{J_s} & -e_0^{J_s} \\ -e_1^{I_s} & e_0^{I_s} \end{pmatrix} \cdot \begin{pmatrix} e_0^{I} & e_0^{J} \\ e_1^{I} & e_1^{J} \end{pmatrix} \\
&= \frac{1}{\frac{1}{f^2}\, q_0^{I_s}q_1^{J_s} - \frac{1}{f^2}\, q_0^{J_s}q_1^{I_s}} \begin{pmatrix} \frac{1}{f}\, q_1^{J_s} & -\frac{1}{f}\, q_0^{J_s} \\ -\frac{1}{f}\, q_1^{I_s} & \frac{1}{f}\, q_0^{I_s} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{f}q_0^{I} & \frac{1}{f}q_0^{J} \\ \frac{1}{f}q_1^{I} & \frac{1}{f}q_1^{J} \end{pmatrix} \\
&= \frac{1}{q_0^{I_s}q_1^{J_s} - q_0^{J_s}q_1^{I_s}} \begin{pmatrix} q_1^{J_s} & -q_0^{J_s} \\ -q_1^{I_s} & q_0^{I_s} \end{pmatrix} \cdot \begin{pmatrix} q_0^{I} & q_0^{J} \\ q_1^{I} & q_1^{J} \end{pmatrix}
\end{aligned} \tag{49}$$

This result is independent of the normalization factor f and the dimension.
The third possible choice of normalization (next to normalizing the basis vectors from both quadrics and normalizing none) is to normalize only the basis vectors from Q^-, since the normalization factor $f = \hat{Q}^-_{22}$ only occurs in its quadric form.
The effect that the normalization of both vectors cancels does only occur when transforming from one local domain into another. For transformations from or into any other coordinate system (e.g., the standard basis) the result still depends on the choice of normalization.
The decision which normalization procedure may be applied is a conceptual one.

3.4.4 Study of the Heatmaps

Figure 5 now shows the calculated heatmap for the following parameters.

$$\begin{gathered}
p = 131 \\
\hat{Q}^+ = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \hat{Q}^- = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{pmatrix} \\
\Rightarrow \mathbf{G}_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{G}_2^{-101}, \qquad f = 8 \\
\hat{e}^0 = \hat{q}^0 = \left\langle \begin{matrix} 1 \\ 0 \\ 1 \end{matrix} \right\rangle \in Q^+ \qquad \text{and} \qquad \hat{e}^1 = \hat{q}^1 = \left\langle \begin{matrix} 0 \\ 56 \\ 1 \end{matrix} \right\rangle \in Q^- \\
\text{(no basis vector is normalized } \equiv \text{ both are normalized)}
\end{gathered} \tag{50}$$

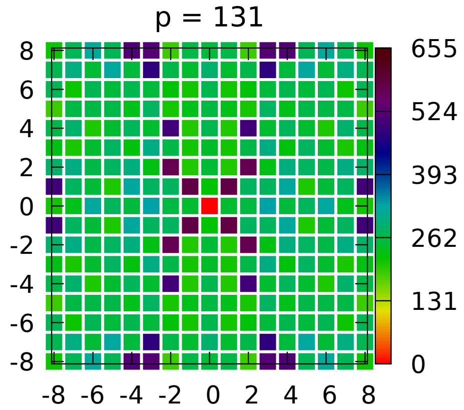


Figure 5: Heatmap H of the selected local world domain in the local coordinate system defined by the basis vectors $\hat{q}^0$ and $\hat{q}^1$ given in Eq. (50). The heatmap represents the local world domain of the selected local coordinate system and the value of each point in the heatmap shows how often this point in the selected local world domain can be successfully transformed into the local world domain of another local coordinate system.

As expected, the heatmap is point symmetric since the equations for the quadric points and the local world domain condition are invariant under a minus sign of the coordinates of a point: $\hat{p}^t\,\hat{Q}^{\pm}\,\hat{p} = (-\hat{p})^t\,\hat{Q}^{\pm}\,(-\hat{p}) = 0$ and $|p_\nu| \leq \sqrt{\frac{p-1}{2}} \Leftrightarrow |-p_\nu| \leq \sqrt{\frac{p-1}{2}}$.
The visible axis symmetry is a special case here, the heatmaps are not axis symmetric in general. The value of the center is zero because it was artificially set to zero (see Section 3.4.2).

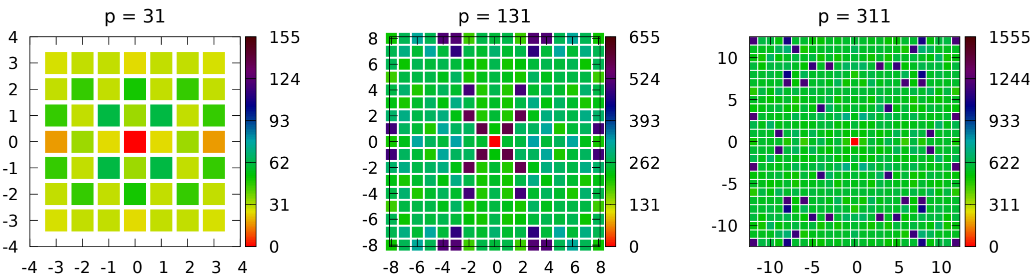
The set of possible local coordinate systems (and therefore the heatmaps) depends on the following factors:

- the prime p
- the biquadric
  - the components of the matrix $\mathbf{G}_2$
  - the effective scaling factor f
- if both or just one basis vector is normalized

The heatmaps also change with the selected local coordinate system that is chosen within the set of all possible local coordinate systems.
Figure 6 shows the heatmaps for different prime numbers p. Since the choice of the prime p changes the field itself, including which numbers are squares and non-squares, also the effective scaling factor f and therefore the biquadric and the selected basis vectors for the selected local world domain change. These factors are given in Eq. (51), $\mathbf{G}_2$ is the same $\mathbf{G}_2 = \mathbf{G}_2^{-101}$ for all

heatmaps.

$$
\begin{aligned}
&p = 31\\
&\qquad f = 11\\
&\qquad \hat{q}^0 = \left\langle \begin{matrix} 1 \\ 0 \\ 1 \end{matrix} \right\rangle \in Q^+ \cap \mathfrak{A}^2 \qquad \text{and} \qquad \hat{q}^1 = \left\langle \begin{matrix} 0 \\ 12 \\ 1 \end{matrix} \right\rangle \in Q^- \cap \mathfrak{A}^2\\
&p = 131\\
&\qquad f = 8\\
&\qquad \hat{q}^0 = \left\langle \begin{matrix} 1 \\ 0 \\ 1 \end{matrix} \right\rangle \in Q^+ \cap \mathfrak{A}^2 \qquad \text{and} \qquad \hat{q}^1 = \left\langle \begin{matrix} 0 \\ 56 \\ 1 \end{matrix} \right\rangle \in Q^- \cap \mathfrak{A}^2\\
&p = 311\\
&\qquad f = 19\\
&\qquad \hat{q}^0 = \left\langle \begin{matrix} 1 \\ 0 \\ 1 \end{matrix} \right\rangle \in Q^+ \cap \mathfrak{A}^2 \qquad \text{and} \qquad \hat{q}^1 = \left\langle \begin{matrix} 0 \\ 35 \\ 1 \end{matrix} \right\rangle \in Q^- \cap \mathfrak{A}^2
\end{aligned}
\tag{51}
$$



(a) $p = 31$ (b) $p = 131$ (c) $p = 311$

Figure 6: Heatmaps H of selected local world domains for different prime numbers p. The parameters are as in Eq. (51).

Not only the size of the local world domain changes (by definition) but also the structure that emerges.
Figure 7 shows the comparison of the heatmap for $p = 131$ above (again shown in Figure 7(c)) with four other heatmaps where for each exactly one parameter was changed from the configuration shown in Eq. (50), except for the prime number $p = 131$.
The first one 7(a) is constructed from a different biquadric, for which the $\mathbf{G}_2$ matrix is a different one $\mathbf{G}_2 = \mathbf{G}_2^{101}$, but the effective scaling factor is the same ($f = 8$).

$$\mathbf{G}_2^{101} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{52}$$

This leads to a new set of possible basis vectors for the local coordinate systems. The basis vectors for the selected local coordinate system are $\hat{q}^0 = \left\langle \begin{matrix} 1 \\ 28 \\ 1 \end{matrix} \right\rangle$ and $\hat{q}^1 = \left\langle \begin{matrix} 0 \\ 56 \\ 1 \end{matrix} \right\rangle$.
The second one 7(b) is also constructed from a different biquadric, but here $\mathbf{G}_2$ is the same as in 7(c), instead, the effective scaling factor is different $f = 10$. This leads to a new set of possible

basis vectors for the local coordinate systems. The ones for the selected local coordinate system are $\hat{q}^0 = \left\langle \begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix} \right\rangle$ and $\hat{q}^1 = \left\langle \begin{smallmatrix} 0 \\ 11 \\ 1 \end{smallmatrix} \right\rangle$.

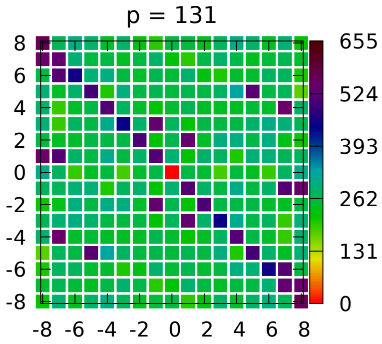
The third one 7(d) is the one where all basis vectors constructed from points in Q^- are normalized. This again leads to a new set of possible local coordinate systems. For all of them, including the selected one, only the second basis vector $\hat{e}^1 = \frac{1}{f}\hat{q}^1$ is different. The new basis vector for the selected local coordinate system is $\hat{e}^1 = \hat{c} + \frac{1}{8}(\hat{q}^1 - \hat{c}) = \left\langle \begin{smallmatrix} 0 \\ 7 \\ 1 \end{smallmatrix} \right\rangle$.

The fourth and last heatmap 7(e) has the same set of possible local coordinate systems as in (c), but the selected local coordinate system is chosen differently.

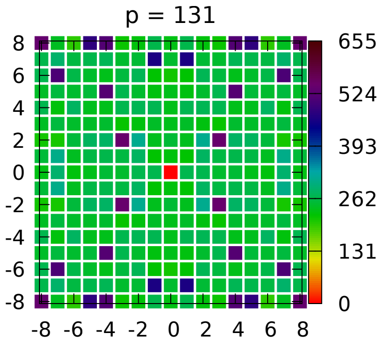
Its basis vectors are $\hat{q}'^0 = \left\langle \begin{smallmatrix} 2 \\ 38 \\ 1 \end{smallmatrix} \right\rangle$ and $\hat{q}'^1 = \left\langle \begin{smallmatrix} 0 \\ 75 \\ 1 \end{smallmatrix} \right\rangle$.

The heatmaps differ in their appearance, but they have some properties in common. The range that holds the values of the heatmaps is the same for all of them. The distribution of the heatmap values is quite homogeneous except for some points that show a significantly higher value, especially when going to higher prime numbers this becomes more and more noticeable. The spatial position of these ‘purple dots’ (the ones having a higher value) does not follow any particular visible pattern, neither in the local world domain of the selected local coordinate system nor in the standard basis.

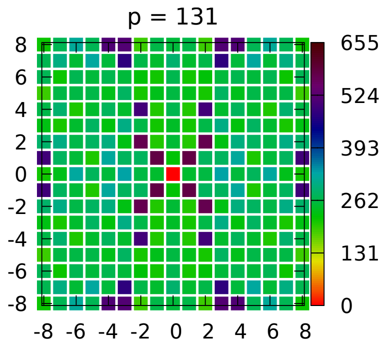
In the following section the occurrence of these ‘purple dots’ and the number distribution of the heatmap values is investigated.



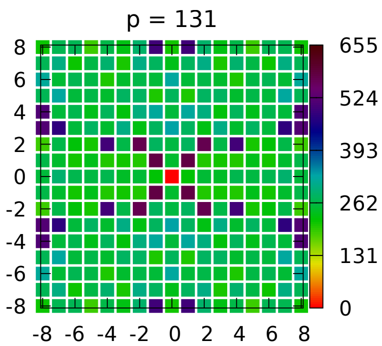
(a) $\mathbf{G}_2 = \mathbf{G}_2^{101}$



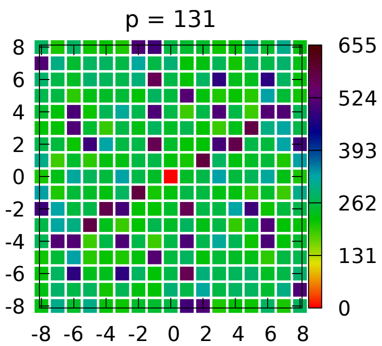
(b) $f = 10$



(c) all parameters are as in Eq. (50)



(d) the second basis vectors of all local coordinate systems are $\hat{e}^1 \neq \hat{q}^1$

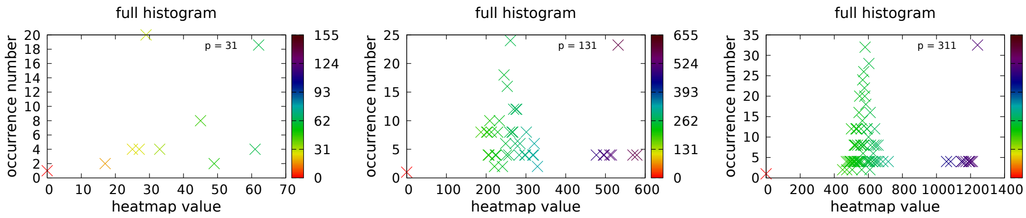


(e) the basis vectors of the selected local coordinate system are $\hat{q}'^0$ and $\hat{q}'^1$

Figure 7: Heatmaps H of selected local world domains for different parameters. If not stated differently, the parameters are as in Eq. (50).

3.4.5 Study of the Occurrence Distribution

To investigate the number distribution (not the distribution in space but in the number of their occurrence) of the heatmap values, a *histogram* is made. It shows the number of points in dependence of their heatmap value.
The histograms for the same three prime numbers p and the same parameters as above (given in Eq. (50)) are shown in Figure 8.

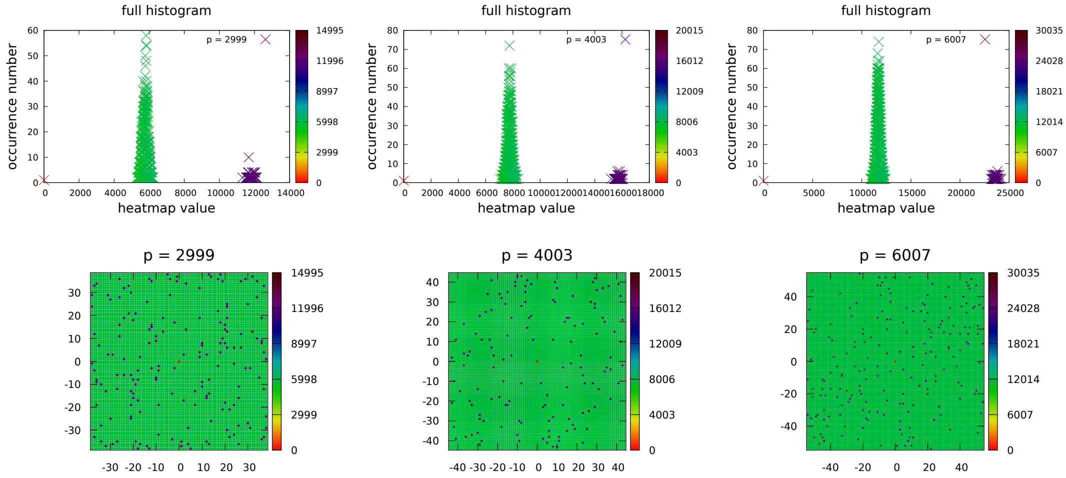


(a) $p = 31$ (b) $p = 131$ (c) $p = 311$

Figure 8: Histograms of the heatmaps H of selected local world domains for different prime numbers p. The histogram shows the number of points in dependence of their heatmap value. The parameters are as in Eq. (50).

Interestingly, two peaks appear which are more pronounced for larger prime numbers p and of which the first one is far larger than the second one.
The first peak corresponds to the large number of points that make up the quite homogeneous distribution of heatmap values, the second smaller peak corresponds to the fewer 'purple dots' seen in the heatmaps.
The point that occurs once and has value zero is the center that was artificially set to zero. For all other heatmap values the number of occurrence has to be a multiple of two, due to the point symmetry of the heatmap.
When going to higher prime numbers the peaks become clearer, better separated and narrower, as one can see in Figure 9.

In the following, the *position*, the *shape*, and the *size* of these peaks is discussed in detail for large prime numbers $p \in \{1999, 2999, 4003, 5003, 6007\}$.
The discussion here focuses on the results given by the biquadric with $\mathbf{G}_2 = \mathbf{G}_2^{101}$ as given in Eq. (52) with the other parameters as in Appendix A.1. The heatmaps produced in these calculations are shown in Appendix A.3.1.
The results for the other biquadric with $\mathbf{G}_2 = \mathbf{G}_2^{-101}$ (and the other parameters as in Appendix A.2) do show analogous behavior and are therefore not discussed in detail here, but given in Appendix A.2.



(a) $p = 1999$ (b) $p = 4003$ (c) $p = 6007$

Figure 9: Histograms and corresponding heatmaps of selected local world domains for large prime numbers p. The parameters for the calculations are given in Appendix A.1 with $\mathbf{G}_2 = \mathbf{G}_2^{101}$.

3.4.5.1 Position Analysis of the Peaks

For all examples that were investigated at different prime numbers larger than about $p \gtrsim 130$, the first peak is at a heatmap point value position of about $2p$, and all values of this peak lie within the range of $1p$ to $3p$. Off the second peak all values are positioned in the range $3p$ to $5p$, centered around about $4p$, which is about twice as much as the center of the first peak.
The maximal value a heatmap point could take in theory (which is also the value the center takes) is the number of local coordinate systems different from the selected one, which is the number of all possible combinations of basis vectors $|Q^+ \cap \mathfrak{A}^2| \cdot |Q^- \cap \mathfrak{A}^2| - 2$. The -2 is for the basis vectors of the selected local coordinate system that are excluded. $|Q^+ \cap \mathfrak{A}^2| = p + 1$ or $p - 1$, depending on the characteristic of the biquadric, the same is true for $|Q^- \cap \mathfrak{A}^2|$. So, depending on the characteristic of the biquadric, there are either $(p + 1)^2 - 2$ or $(p - 1)^2 - 2$, i.e., $\propto p^2$, different local coordinate systems that are not the selected one.
When defining the mean value of a peak μ as just the arithmetic mean of the heatmap values of all points within this peak, one can look at the fraction of the mean divided by the prime number $\frac{\mu}{p}$ of the peak. A value is said to be within the first and second peak if it is within the range $[1p, 3p]$ and $[3p, 5p]$, respectively. Table 1 shows these fractions $\frac{\mu}{p}$ for different prime numbers p. These values confirm the observation that the first peak is centered around $2p$ and the second one around $4p$.
Interestingly, for the prime $p = 5003$ where the plots for the shape of the peaks (see below) show a particular clear pattern the mean is very close to $2p$ and $4p$.
However, with further increasing prime number p the mean does not come closer to $2p$ and $4p$, respectively, it even becomes smaller again.

Remember, that for each transformation from a local coordinate system with current basis vectors $\hat{e}_0^I$ and $\hat{e}_1^J$, also the current coordinate systems with basis vectors $\hat{e}_0^I$ and $-\hat{e}_1^J$, $-\hat{e}_0^I$ and $\hat{e}_1^J$, and $-\hat{e}_0^I$ and $-\hat{e}_1^J$ do exist in the set of all possible basis vectors for current local coordinate systems. These, however, give the same current local coordinate system, only reflected at the axes and point reflected at the center, respectively. Therefore, one has four different selected local coordinate systems that give, in number, not in position of the heatmap values, exactly the same contribution.

prime p	$\frac{\mu}{p}$ of the first peak	$\frac{\mu}{p}$ of the second peak
1999	1.92	3.90
2999	1.93	3.93
4003	1.93	3.92
5003	2.00 (1.99611)	3.99
6007	1.94	3.93

Table 1: Mean value of the peak μ divided by the prime number p for different primes p. The parameters for the calculations are given in Appendix A.1 with $\mathbf{G}_2 = \mathbf{G}_2^{101}$.

3.4.5.2 Size Analysis of the Peaks

Now, Table 2 shows the percentage of the points that are in the second peak for the same primes as above.

prime p	number of points in the second peak	number of all points	percentage of points in the second peak
1999	110	3968	2.77 %
2999	170	5928	2.87 %
4003	136	7920	1.72 %
5003	216	10200	2.12 %
6007	196	11880	1.65 %

Table 2: Number of points in the second peak, number of all points and percentage of the points in the second peak for different primes p. The parameters for the calculations are given in Appendix A.1 with $\mathbf{G}_2 = \mathbf{G}_2^{101}$.

The percentage of points within the second peak changes a bit for different prime numbers, but it is similarly small for all these.

3.4.5.3 Shape Analysis of the Peaks

To further investigate the shape of the peaks the peaks are investigated separately.
Since the number of points in the second peak is very small (far smaller than the points in the first peak) it is not in detail analyzed for its shape. To do this one would have to go to even higher prime numbers, and therefore even more time consuming calculations (the calculation time grows about quadratic with the prime number). However, one plot of the normalized second peak (all values are divided by the sum of all values within the peak) for the prime number $p = 7019$ is shown in Figure 10, the parameters for this calculation are given in Appendix A.1. The small number of points does not allow for a reasonable detailed analysis of the peak shape but when comparing with the first peak shown in Figure 11 (see below for the discussion of these plots) one could assume that the shape of the second peak has similar features as the first peak.

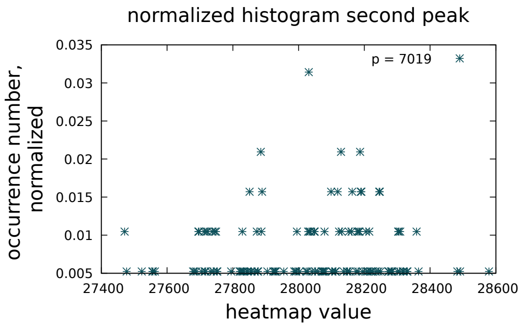


Figure 10: The normalized second peak of the histogram for $p = 7019$. The small number of points does not allow for a reasonable detailed analysis of the peak shape.
The parameters for the calculations are given in Appendix A.1 with $\mathbf{G}_2 = \mathbf{G}_2^{101}$.

Figure 11 shows the first peak for the same prime numbers p as above processed in two different ways.
To obtain the plot on the *left side* the peak is normalized (all values are divided by the sum of all values). In addition, the mean μ, that is calculated as the arithmetic mean of all points in the peak as in Section 3.4.5.1, is marked.
The structure that emerges suggests that the peak is of the shape of a Gaussian distribution, with some additional constant contribution at low values. Especially for $p = 5003$ this is clearly visible, there is a 'window' below the Gaussian, separating it from the constant contribution at low values.
This was further tested by fitting a Gaussian distribution of the form shown in Eq. (53) to the peak. The mean in the Gaussian is the calculated mean μ and the standard deviation σ is assumed to be proportional to $\sqrt{p}$. The factor a, $\sigma = a \cdot \sqrt{p}$, is the fit parameter.

$$f(x) = \frac{1}{\sqrt{2\pi(a\sqrt{p})^2}} \exp\Big(- \frac{(x-\mu)^2}{2(a\sqrt{p})^2}\Big) \tag{53}$$

The fitted Gaussian lies below the curve of the data since the points at low values also play into the normalization of the data peak.
The value a is shown in Table 3, and it is quite similar for all p that were investigated. This shows that the standard derivative is only proportional to $\sqrt{p}$, therefore the peaks are getting narrower for increasing prime number p.

prime p	$a = \frac{\sigma}{\sqrt{p}}$
1999	2.208 ± 0.134
2999	2.209 ± 0.124
4003	2.157 ± 0.116
5003	2.280 ± 0.100
6007	2.192 ± 0.103

Table 3: Fit parameter a of the Gaussian distribution as given in Eq. (53) with mean μ and standard deviation $\sigma = a \cdot \sqrt{p}$ for different primes p. The parameters for the calculations are given in Appendix A.1 with $\mathbf{G}_2 = \mathbf{G}_2^{101}$.

Another test for the Gaussian shape of the peak is shown on the *right side* of Figure 11.

For these plots all values that are part of the first peak are depicted with the x axis being $\frac{(x-\mu)^2}{p}$ and the y axis being in logarithmic scale. In this depiction a Gaussian peak would appear to be a linear function in x, see Eq. (54).
A Gaussian distribution with mean μ and standard deviation σ is of the form

$$\begin{aligned} y = f(x) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2}) \\ \ln(y) &= -\frac{1}{2\sigma^2}(x-\mu)^2 - \ln(\sqrt{2\pi\sigma^2}) \\ \Rightarrow \ln(y) \ &\propto \ -(x-\mu)^2 \ + \ const. \end{aligned} \tag{54}$$

The plots show a linear trend for the points that are not part of the constant contribution at low values, and therefore emphasize the suggestion of the distribution being Gaussian-like.

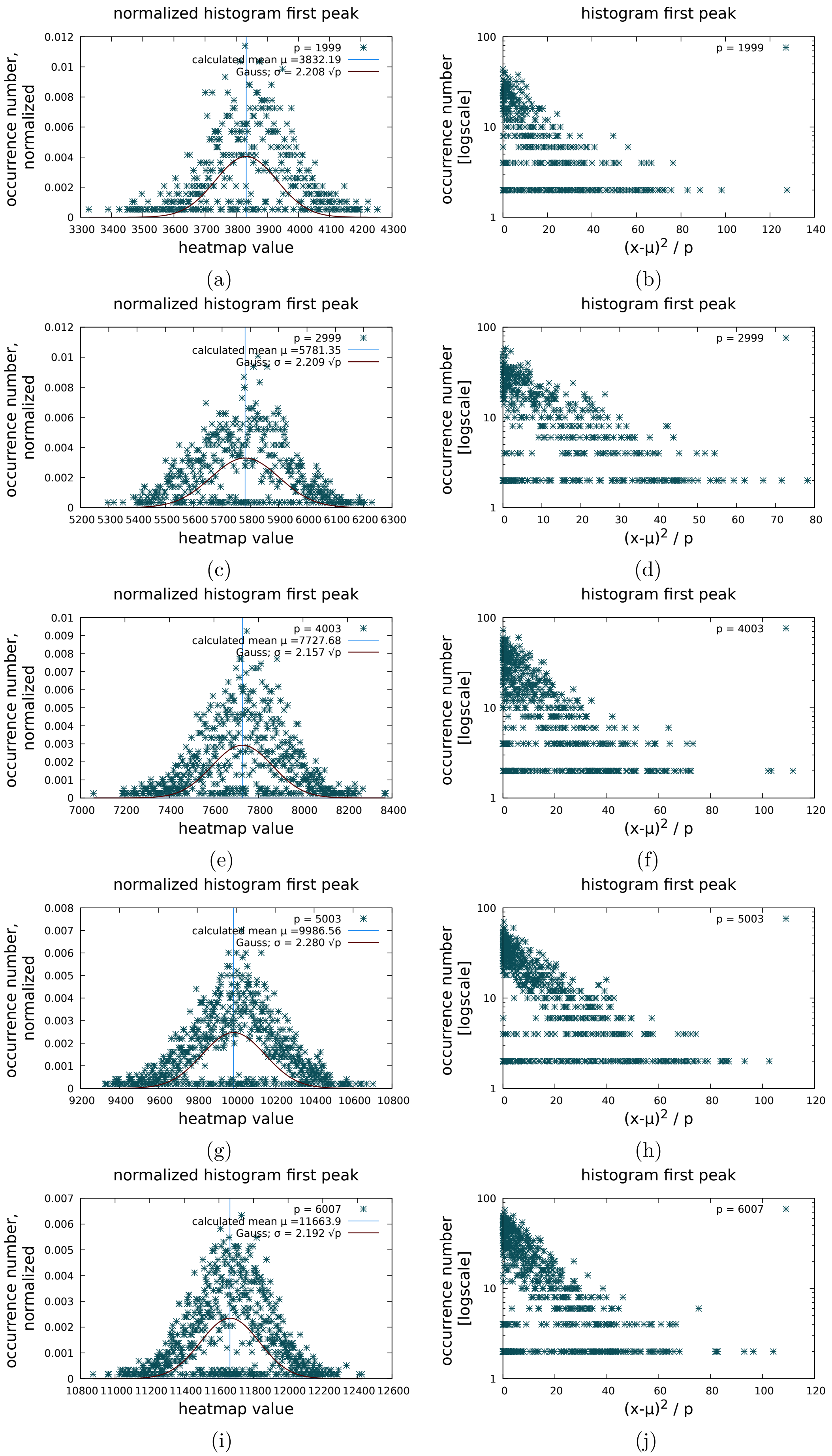


Figure 11: The first peak of the histogram for different prime numbers p.
left: The normalized peak, with the calculated mean ν and a Gaussian distribution as given in Eq. (53) fitted to it. *right:* The x axis is $\frac{(x-\mu)^2}{p}$ and the y axis is in logarithmic scale.
The parameters for the calculations are given in Appendix A.1 with $\mathbf{G}_2 = \mathbf{G}_2^{101}$.

3.4.5.4 Stability Tests

Last, the stability of these findings w.r.t. the parameters changing the heatmaps is tested. Since the discussion above was made for different primes p the findings can be considered as stable w.r.t. p (at least within the investigated range).
The other parameters having influence on the heatmaps were

- the biquadric, i.e., $\mathbf{G_d}$ and f
- the normalization choice
- the choice of new basis vectors for the selected local coordinate system

Now, calculate and plot, respectively, the above investigated properties of the peaks (position, size and shape) for a change in these parameters.
Since the calculations for large prime numbers are quite time consuming, the parameters will be changed to only one other value to compare with the results above. So, no scan of the whole 4-dimensional parameter space is done, instead from the one point in the parameter space describing the parameters used above, only one step is taken in one direction each time. Furthermore, one time all parameters are changed.
The medium-large (within the investigated range) prime number $p = 2999$ was chosen to combine meaningful results with reasonable calculation times.

The parameters are separately replaced by these values

$$\mathbf{G}_2 = \mathbf{G}_2^{-101} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{this yields } \hat{q}'^0 = \left\langle \begin{matrix} 1 \\ 0 \\ 1 \end{matrix} \right\rangle \quad \text{and} \quad \hat{q}'^1 = \left\langle \begin{matrix} 0 \\ 219 \\ 1 \end{matrix} \right\rangle \tag{55}$$

$$f = 29, \quad \text{this yields } \hat{q}'^0 = \left\langle \begin{matrix} 4 \\ 1304 \\ 1 \end{matrix} \right\rangle \quad \text{and} \quad \hat{q}'^1 = \left\langle \begin{matrix} 0 \\ 697 \\ 1 \end{matrix} \right\rangle \tag{56}$$

$$\hat{e}^1 = \frac{1}{f}\hat{q}^1 = \left\langle \begin{matrix} 0 \\ 1835 \\ 1 \end{matrix} \right\rangle \tag{57}$$

$$\hat{q}'^0 = \left\langle \begin{matrix} 4 \\ 1695 \\ 1 \end{matrix} \right\rangle \quad \text{and} \quad \hat{q}'^1 = \left\langle \begin{matrix} 0 \\ 2780 \\ 1 \end{matrix} \right\rangle. \tag{58}$$

1. **Peak Position**

parameter choice	$\frac{\mu}{p}$ of the first peak	$\frac{\mu}{p}$ of the second peak
as above	1.93	3.93
$\mathbf{G}_2$ as in Eq. (55)	1.93	3.92
f as in Eq. (56)	1.93	3.93
basis vector from Q^- normalized as in Eq. (57)	1.93	3.93
choice for the basis vectors for local COS as in Eq. (58)	1.93	3.93
all parameters different, as in Eqs. (55), (56), (57), and (58)	1.93	3.93

Table 4: Mean value of the peak μ divided by p for different parameter choices. If not stated differently, the parameters are as in the 'as above' case which are given in Appendix A.1 with $\mathbf{G}_2 = \mathbf{G}_2^{101}$.

Note, that the values are only the same when rounded, they are not exactly the same. However, the values differ very little from each other, so the mean value of the peaks is very stable w.r.t. the changing parameters.

2. **Peak Size**

parameter choice	number of points in the second peak	number of all points	percentage of points in the second peak
as above	170	5928	2.87 %
$\mathbf{G}_2$ as in Eq. (55)	136	5928	2.29 %
f as in Eq. (56)	184	5928	3.10 %
basis vector from Q^- normalized as in Eq. (57)	170	5928	2.87 %
choice for the basis vectors for local COS as in Eq. (58)	170	5928	2.87 %
all parameters different, as in Eqs. (55), (56), (57), and (58)	190	5928	3.21 %

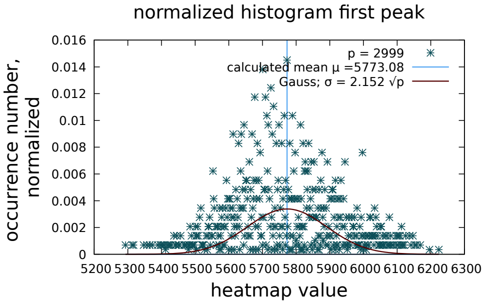
Table 5: Number of points in the second peak, number of all points and percentage of the points in the second peak for different parameter choices. If not stated differently, the parameters are as in the 'as above' case which are given in Appendix A.1 with $\mathbf{G}_2 = \mathbf{G}_2^{101}$.

Also the size of the peaks and the percentage of points in the second peak, respectively, is quite stable w.r.t. the changing parameters.

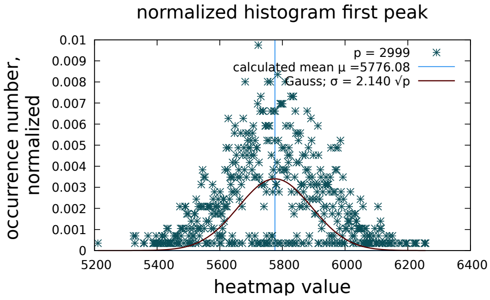
3. **Peak Shape**

Looking at the plots given in Figures 12 and 13, the form of the first peak is quite stable. Mostly the shape of the first peak for a different $\mathbf{G}_d = \mathbf{G}_d^{-101}$ differs from the others, because the separation of the upper Gaussian distribution and the lower constant part is not clearly visible.

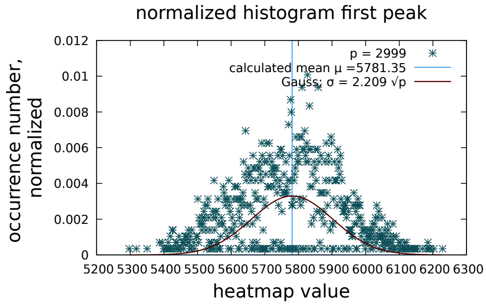
The peak shapes of the first peak for a different $\mathbf{G}_d = \mathbf{G}_d^{-101}$ were also investigated at other prime numbers and can be seen in Appendix A.2. The plots show that when going to higher prime numbers than $p = 2999$ the separation of the different contributions and the Gaussian shape become visible.



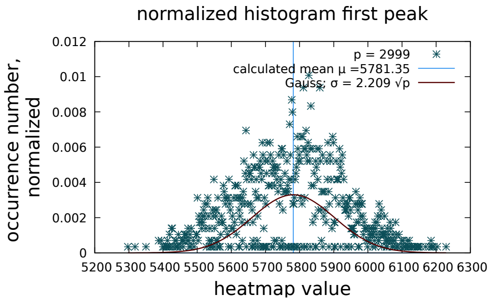
(a) $\mathbf{G}_2$ as in Eq. (55)



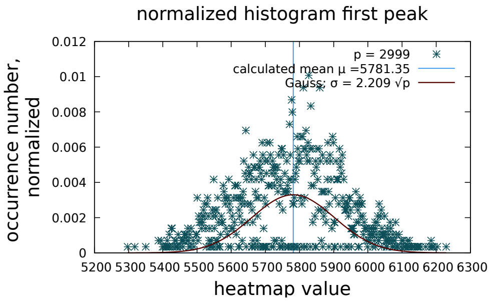
(b) f as in Eq. (56)



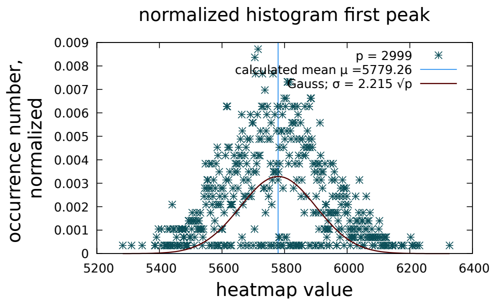
(c) as above



(d) basis vector from Q^- normalized as in Eq. (57)

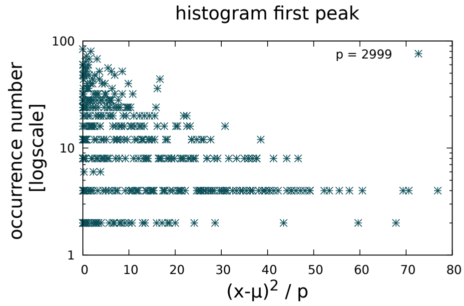


(e) choice for the basis vectors for local COS as in Eq. (58)

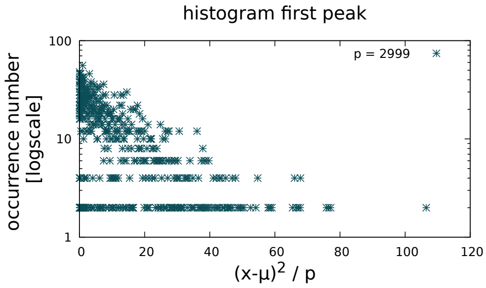


(f) all parameters different, as in Eqs. (55), (56), (57), and (58)

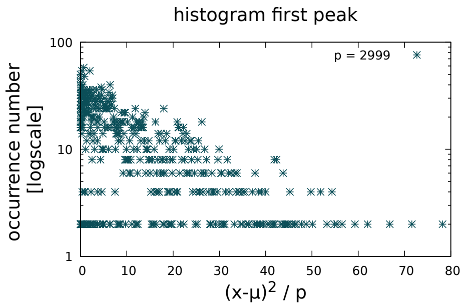
Figure 12: The first peak of the histogram, normalized, with the calculated mean ν and a Gaussian distribution as given in Eq. (53) fitted to it for different parameter choices. If not stated differently, the parameters are as in the 'as above' case (c) which are given in Appendix A.1 with $\mathbf{G}_2 = \mathbf{G}_2^{101}$.



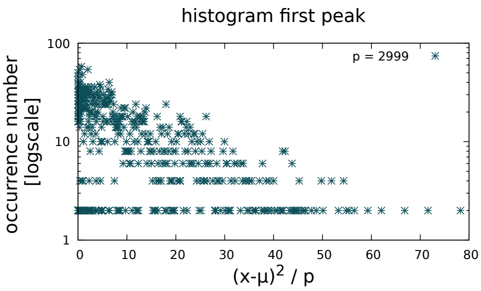
(a) $\mathbf{G}_2$ as in Eq. (55)



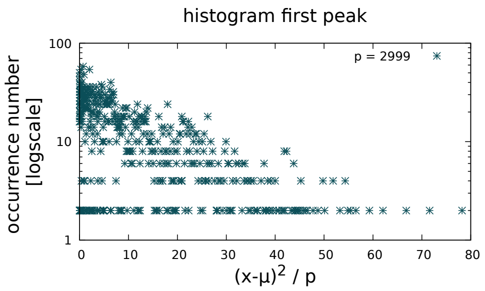
(b) f as in Eq. (56)



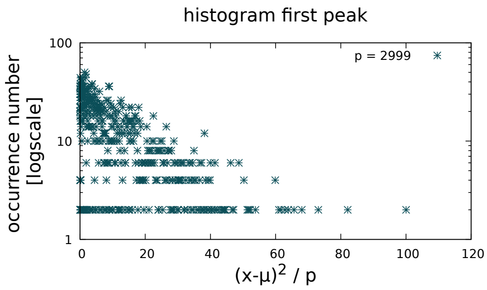
(c) as above



(d) basis vector from Q^- normalized as in Eq. (57)



(e) choice for the basis vectors for local COS as in Eq. (58)



(f) all parameters different, as in Eqs. (55), (56), (57), and (58)

Figure 13: The first peak of the histogram, the x axis is $\frac{(x-\mu)^2}{p}$ and the y axis is in logarithmic scale for different parameter choices. If not stated differently, the parameters are as in the 'as above' case (c) which are given in Appendix A.1 with $\mathbf{G}_2 = \mathbf{G}_2^{101}$.

So, for all parameters the results are quite stable when changing to another parameter value.

3.5 Summary

After introducing and motivating the definition of a local world domain an example of a local world domain is shown in the standard basis. This example elucidates the conclusion that it is indeed possible for biquadric points (that are different from the ones chosen to serve as new basis vectors for the selected local coordinate system) to be within the local world domain. So, the prior assumption that this is not possible has to be dropped.

The number of possible transformations that can successfully transform a point within the local world domain of a selected local coordinate system into the local world domain of any other possible local coordinate system is investigated by the use of heatmaps. The points of such a heatmap describe the number of successful transformations into the local world domain of other local coordinate systems.
The possible choices of local coordinate systems depend on a number of parameters. These are the prime number p, from the biquadric $\mathbf{G}_d$ and f, and the choice of normalization for the basis vectors. These and the choice of basis vectors for the selected local coordinate system influence the heatmap.
Nevertheless, there are also some properties that are common to all possible heatmaps, these are studied with the use of histograms depicting the occurrence number of the heatmap values. For most points the number how often they can be transformed into other local world domains is around $2p$, following a Gaussian-similar distribution centered at this mean of $\sim 2p$.
For few ($\lesssim 3\%$) of all points, the 'purple dots', the number they can be successfully transformed into other local world domains is twice as large, following a distribution centered around $\sim 4p$. The exact shape of this distribution is yet unknown, because these few points allow only a small statistic, it might be Gaussian-like as well.
These findings are (within the investigated range) stable w.r.t. another choice of the parameters that do change the heatmaps and histograms.

4 The Projective Space Equipped with Biquadrics as a Graph

In this part the finite projective geometry with its neighbouring relations induced by a biquadric field will be viewed as a graph. The goal is to use the tools of graph theory to further analyze the structure of the geometry.

4.1 The Graph Generated from a Projective Space Equipped with Biquadrics

One can generate a graph from a finite projective geometry that is equipped with a general biquadric field by the following steps:

- the points in the geometry are the *vertices* of the graph
- for a point add *edges* from this point to all points that are within the biquadric attached to this point (its neighbours)
- do this with all other points, too

For a general biquadric field it is that $\hat{p}_1 \in Q^{\pm}_{\hat{p}_2} \nRightarrow \hat{p}_2 \in Q^{\pm}_{\hat{p}_1}$, i.e., the neighbours of a point do not necessarily have to have this point as a neighbour, too. So, the graph generated from this biquadric field has to be a *directed* one.
Some of the properties this generated graph has are listed in the following [Estrada and Knight, 2015] [Bondy and Murty, 1976]:

- finite
  A graph is *finite* if both vertex set and edge set are finite.
- directed
  A *directed graph* is a ordered triple, consisting of a nonempty set of vertices, a disjoint set of arcs and an incidence function that associates with each arc an ordered pair of vertices. Therefore, the edges, also called arcs, are directed [Bondy and Murty, 1976].
- simple
  A directed graph is *simple* if it has no loops (edges going to to the same vertex they are coming from) and no two edges going from the same vertex to the same other vertex.
- When viewing the graph as a weighted one, each edge has the same weight of one.

One special case is a biquadric field that is generated by the translation of one biquadric to all points. (For the form of such a translated (bi)quadric see below in Section 4.2.1.)
This case has the special property that the neighbours of a point do have this point as a neighbour, too, i.e., $\hat{p}_1 \in Q^{\pm}_{\hat{p}_2} \Leftrightarrow \hat{p}_2 \in Q^{\pm}_{\hat{p}_1}$. So, the graph generated from this biquadric field is an *undirected* one.
Some of the properties the graph has when restricting it to the affine space by taking the subgraph of all vertices (points) that are within the affine space are:

- finite
  As above.
- undirected
  All edges are undirected.

◇ simple
  A undirected graph is *simple* if it has no loops and no edges joining the same pair of vertices.

◇ regular
  A graph is *k-regular* if the number of edges at one vertex $d(v) = k$ is k for all vertices[18]. Here, $k = |Q^{\pm} \cap \mathfrak{A}^d|$, the number of affine points within $Q^{\pm}$.

◇ connected
  A graph is *connected* if it has exactly one component, i.e., exactly one subset for which all vertices are connected[19].

◇ When viewing the graph as a weighted one, each edge has the same weight of one.

The *diameter diam* of a graph is defined as the maximal distance between all vertex combinations, where the distance between two vertices is defined as the length of the shortest path. The length of a path is the number of edges one has to pass on the way from one vertex to the other.

Another concept that will be used later is the one of isomorphy of two graphs [Rahman, 2017]. Two graphs G and H are isomorphic if there are bijections, one between the sets of vertices $\theta : V(G) \to V(H)$ and one between the sets of edges $\phi : E(G) \to E(H)$, such that the vetices u and v are connected if and only if $\theta(u)$ and $\theta(w)$ are connected, i.e., their pair is within the set of edges $E(H) = \phi(E(G))$.
Applied to a projective space a *projective transformation* (projectivity) was a non-degenerate linear transformation that conserves incidence relations (see Section 2.2).
These projectivities therefore also conserve quadrics and biquadrics, and, applied to the graph view, biquadrics that can be transformed into each other by means of projectivities result in isomorphic graphs.

Hence, for investigating the structure of the graph built from the finite projective geometry equipped with a translated biquadric field it is sufficient to consider the few different types of biquadrics.

4.2 Diameter of the Graph for Translated Quadrics

In the following now it is proven that the diameter *diam* of a graph that is generated from the finite field equipped with the same quadric translated to every point, restricted to the affine space, is quite small by being $diam = 2$ in the general case with dimension $d \geq 3$ and $diam \leq 3$ in dimension $d = 2$.

Since the proposition can even be proven using only a quadric field instead of a biquadric field, the following considerations will be restricted to *quadric fields*.

The idea is to first find the form of a quadric point of the original quadric, call this point ‘first’ quadric point. Then, the original quadric is translated to one of these ‘first’ quadric points and one has to find the general form of the points within this translated quadric, called ‘second’

[18]The directed graph for the general non-translated case is not necessarily regular, since for regular graphs the requirement for being directed includes also that the number of ingoing edges has to be the same as the number of outgoing edges. This is not fulfilled in general there.

[19]For the directed graph in the general non-translated case it is not as easy to see that it is connected but most certainly it is.

quadric points. All ‘second’ quadric points are now two steps away from the center of the original quadric. If these ‘second’ quadric points can cover all possible points within the affine space, the diameter of the generated graph is $diam = 2$.

4.2.1 Form of a Translated Quadric

First, investigate the form of the original quadric translated to a ‘first’ quadric point and the form of the ‘second’ quadric points within this translated quadric.

The quadric matrix $\hat{Q}_{\hat{p}'}$ of the quadric $Q_{\hat{p}'}$ translated to a point $\hat{p}'$ is of the form [Richter-Gebert and Orendt, 2009]

$$\hat{Q}_{\hat{p}'} = (T_{\hat{p}'}^{-1})^t \ \hat{Q} \ T_{\hat{p}'}^{-1} \tag{59}$$

with

$$T_{\hat{p}'} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & p'_0 \\ 0 & 1 & 0 & \dots & 0 & p'_1 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 1 & p'_{d-1} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}. \tag{60}$$

Now, find a form for the points $\hat{p}''$ that are within this translated quadric, i.e., the ones that fulfill $\hat{p}''^t \ \hat{Q}_{\hat{p}'} \ \hat{p}'' = 0$.

$$\begin{aligned} \hat{p}''^t \ \hat{Q}_{\hat{p}'} \ \hat{p}'' &= \hat{p}''^t \ \left((T_{\hat{p}'}^{-1})^t \ \hat{Q} \ T_{\hat{p}'}^{-1} \right) \hat{p}'' \\ &= (T_{\hat{p}'}^{-1} \ \hat{p}'')^t \ \hat{Q} \ (T_{\hat{p}'}^{-1} \ \hat{p}'') \\ &\overset{!}{=} 0 \\ &= \hat{p}^t \ \hat{Q} \ \hat{p} \qquad \text{for } \hat{p} \text{ a quadric point of } \hat{Q}. \end{aligned} \tag{61}$$

So, if one chooses the ‘second’ quadric point to be

$$\hat{p}'' = T_{\hat{p}'} \ \hat{p} \ \Leftrightarrow \ \hat{p} = T_{\hat{p}'}^{-1} \ \hat{p}'' \tag{62}$$

one obtains that it is indeed a quadric point to the translated quadric

$$\hat{p}''^t \hat{Q}_{\hat{p}'} \hat{p}'' = 0. \tag{63}$$

Therefore, a ‘first’ quadric point, a quadric point $\hat{p}$ to the un-translated quadric, translated by another (or the same) ‘first’ quadric point $\hat{p}'$ yields a quadric point $\hat{p}''$ to the translated quadric $Q_{\hat{p}'}$.

4.2.2 Graph Diameter in Dimension $d \geq 3$

Proposition. A graph generated by the translation of a quadric in the affine space has diameter $diam = 2$ for dimension $d \geq 3$.
The special cases $d = 2$ is handled below.

Proof.
The outline of the proof is as follows: first, one wants to get the form of any ‘second’ quadric point, i.e., any point that is away two steps from the center, and then show that its coordinates

can be chosen as any number without any restriction and therefore that any point can be this ‘second’ quadric point, i.e., two steps away from the initially chosen point.

The proof has to be done separately for even and odd dimensions.
Within each dimension class it is sufficient to show the correlation for the one (in even dimensions) and two (in odd dimensions), respectively, invariant types of quadric forms[20] since the result can be transferred w.l.o.g. to all quadrics by means of projectivities and the diameter of two isomorphic graphs is the same.
Choosing the center $\hat{c} = \langle 0, 0, ..., 0, 1 \rangle^t$ as the starting point for these two steps is no loss of generality since the quadric is the same, just translated, in every point, so the graph looks the same from the viewpoint of every point.

In Even Dimensions

Starting in *even dimensions* the quadric is chosen to be of the form as in Eq. (16) above in Section 2.3 [21]

$$\hat{Q} = \begin{pmatrix} B & O & ... & O & \vec{0} \\ O & B & ... & O & \vec{0} \\ & & ... & & \\ O & O & ... & B & \vec{0} \\ \vec{0}^t & \vec{0}^t & ... & \vec{0}^t & 2 \end{pmatrix}, \tag{64}$$

where B is a 2×2 matrix with zeros on the diagonal, and ones on the off-diagonals, while O is the 2×2 zero matrix.

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{65}$$

Since the matrix $\hat{Q}$ is invertible ($\det(\hat{Q}) = (-1)^{d/2}\ 2 \neq 0$) and symmetric, it is indeed a representation of a quadric. Furthermore, it preserves the affine space.
With a matrix of this form one has the equation for the points $\hat{p}$ within the quadric as

$$\begin{aligned} \hat{p}^t \hat{Q} \hat{p} &= \sum_{i=0}^{d} \sum_{j=0}^{d} \ p_i \ \hat{Q}_{ij} \ p_j \\ &= 2p_0p_1 + 2p_2p_3 + ... + 2p_{d-2}p_{d-1} + 2 \\ &= 2 + \sum_{i=0}^{\frac{d-2}{2}} \ 2 \ p_{2i} \ p_{2i+1} \\ &= 2 + 2p_0p_1 + \sum_{i=1}^{\frac{d-2}{2}} \ 2 \ p_{2i} \ p_{2i+1} \\ &\stackrel{!}{=} 0. \end{aligned} \tag{66}$$

[20] In even dimensions all quadrics are projectively invariant to the form $\hat{P}$ (parabolic), while in odd dimensions all quadrics are projectively invariant either to the form $\hat{H} = \hat{F}_1$ or $\hat{E} = \hat{F}_{\bar{q}}$ (hyperbolic and elliptic). See also Section 2.3.

[21] Here, w.l.o.g., the normalization of the quadric is chosen such that $\hat{Q}_{dd} = 2$ (and not such that $\hat{Q}_{dd} = -1$ as in Eq. (11) in Section 2.3 above).

Taking this equation as the restriction for the $\hat{p}_0$ component and having arbitrary parameters $\hat{p}_i$ for the other components, one has the following form for the quadric points

$$\hat{p} = \begin{pmatrix} -\frac{1+\sum_{i=1}^{\frac{d-2}{2}} p_{2i}\ p_{2i+1}}{p_1} \\ p_1 \\ p_2 \\ \dots \\ p_{d-1} \\ 1 \end{pmatrix}, \tag{67}$$

with $p_i \in \mathbb{F}$ for $i \geq 2$ and $p_1 \in \mathbb{F}\backslash\{0\}$. So, the only parameter restriction is $p_1 \neq 0$.

To get the form of any point $\hat{p}''$ on any 'second' quadric (the generic form of a quadric point of the first quadric points), it is sufficient to do a translation of a general 'first' quadric point $\hat{p}$ by another (or the same) 'first' quadric point $\hat{p}'$, as shown above in Section 4.2.1.
Using Eq. (67) for the form of a 'first' quadric point for $\hat{p}$ and $\hat{p}'$ one obtains the form of a general 'second' quadric point $\hat{p}''$ as

$$\begin{aligned} \hat{p}'' = T_{\hat{p}'}\ \hat{p} &= \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & -\frac{1+\sum_{i=1}^{\frac{d-2}{2}} p'_{2i}\ p'_{2i+1}}{p'_1} \\ 0 & 1 & 0 & \dots & 0 & p'_1 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 1 & p'_{d-1} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1+\sum_{i=1}^{\frac{d-2}{2}} p_{2i}\ p_{2i+1}}{p_1} \\ p_1 \\ \dots \\ p_{d-1} \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1+\sum_{i=1}^{\frac{d-2}{2}} p_{2i}\ p_{2i+1}}{p_1} - \frac{1+\sum_{i=1}^{\frac{d-2}{2}} p'_{2i}\ p'_{2i+1}}{p'_1} \\ p_1 + p'_1 \\ p_2 + p'_2 \\ \dots \\ p_{d-1} + p'_{d-1} \\ 1 \end{pmatrix}. \end{aligned} \tag{68}$$

The only other parameter restrictions (next to $p_1 \neq 0$) is $p'_1 \neq 0$.
Now, show that this 'second' quadric point can be any point in the affine space.

For this choose, w.l.o.g., $p'_i = 0$ for $4 \leq i \leq d-1$ since the remaining parameters are sufficient for the proof.
So,

$$\hat{p}'' = \begin{pmatrix} -\frac{1+\sum_{i=1}^{\frac{d-2}{2}} p_{2i}\ p_{2i+1}}{p_1} - \frac{1+p'_2 p'_3}{2p'_1} \\ p_1 + p'_1 \\ p_2 + p'_2 \\ p_3 + p'_3 \\ p_4 \\ \dots \\ p_{d-1} \\ 1 \end{pmatrix}. \tag{69}$$

Thus, one has the following system of equations

$$\begin{aligned}
p_0'' &= -\frac{1+\sum_{i=1}^{\frac{d-2}{2}} \ p_{2i} \ p_{2i+1}}{p_1} - \frac{1+p_2'p_3'}{p_1'} && (70)\\
p_j'' &= p_j + p_j' \quad \text{for } 1 \le j \le 3 \\
&\Leftrightarrow p_j = p_j'' - p_j' \quad \text{for } 1 \le j \le 3 && (71)\\
p_i'' &= p_i \quad \text{for } 4 \le i \le d-1 \ . && (72)
\end{aligned}$$

Defining $\sum_{i=1}^{\frac{d-2}{2}} \ p_{2i} \ p_{2i+1} =: \Sigma$ and inserting the form of p_1 in Eq. (71) into the equation for p_0'' one has

$$\begin{aligned}
p_0'' &= -\frac{1+\Sigma}{p_1''-p_1'} - \frac{1+p_2'p_3'}{p_1'} \\
\Leftrightarrow \ 0 &= p_0''(p_1''-p_1')p_1' + (1+\Sigma)p_1' + (1+p_2'p_3')(p_1''-p_1') \\
0 &= \Big(-p_0''\Big) \ p_1'^2 + \Big(p_0''p_1'' + (1+\Sigma) - (1+p_2'p_3')\Big) \ p_1' + \Big((1+p_2'p_3')p_1''\Big) \\
&= a \ p_1'^2 + b \ p_1' + c \ . && (73)
\end{aligned}$$

This equation now has the form of a quadratic equation in p_1'. It is solvable if the discriminant $D = b^2 - 4ac$ is a square number[22]. This is fulfilled, e.g., if $a = 0$ or $c = 0$.
First, consider the case $p_0'' \neq 0$ and $p_1'' \neq 0$. There the condition $c = 0$ can be met by choosing the free parameters p_2' and p_3' to fulfill

$$\begin{aligned}
p_2'p_3' &= -1 && (74)\\
&\Leftrightarrow c = (1+p_2'p_3')p_1'' = (1-1) \ p_1'' = 0.
\end{aligned}$$

Since $p_1' \overset{!}{\neq} 0$, one here also has the requirement $b \neq 0$, which is also fulfilled.
From Eq. (74) it follows that

$$\begin{aligned}
p_0' &= -\frac{1-1}{p_1'} = 0 \\
&\Rightarrow p_0'' = -\frac{1+\Sigma}{p_1} + 0 \\
&\Rightarrow (1+\Sigma) = -p_0''p_1 \ . && (75)
\end{aligned}$$

So,

$$\begin{aligned}
b &= p_0''p_1'' + (1+\Sigma) - (1+p_2'p_3') \\
&= p_0''p_1'' - p_0''p_1 - (1-1) \\
&= p_0''(p_1''-p_1) \\
&= p_0''p_1' \neq 0 && (76)
\end{aligned}$$

since $p_0'' \neq 0$ and $p_1' \neq 0$.

[22]This is also true in Galois field if the prime p is odd, i.e., $p > 2$ [Moorhouse, 2007].

In the special case $p_0'' = 0$, but $p_1'' \neq 0$, it follows that $a = 0$ and the quadratic equation in Eq. (73) becomes a linear one.

$$0 = \Big((1+\Sigma) - (1 + p_2' p_3') \Big) \; p_1' + \Big((1 + p_2' p_3') p_1'' \Big) \tag{77}$$

This equation can be solved by

$$p_1' = - \frac{(1 + p_2' p_3') p_1''}{(1+\Sigma) - (1 + p_2' p_3')} \; . \tag{78}$$

This requires to choose $p_2' p_3' \neq \Sigma$ (to not divide by zero) and $p_2' p_3' \neq -1$ (since $p_1' \overset{!}{\neq} 0$). However, it is still possible to choose $p_2' p_3'$ such that it fulfills both.

In the opposite special case $p_1'' = 0$, but $p_0'' \neq 0$, it follows that $c = 0$ and the quadratic equation becomes also a linear one when dividing by p_1'.

$$0 = \Big(-p_0'' \Big) \; p_1' + \Big((1+\Sigma) - (1 + p_2' p_3') \Big) \tag{79}$$

This equation can be solved by

$$p_1' = \frac{(1+\Sigma) - (1 + p_2' p_3')}{p_0''} \; . \tag{80}$$

This also requires to choose $p_2' p_3' \neq \Sigma$ (since $p_1' \overset{!}{\neq} 0$).

The last special case, where $p_0'' = 0$ and $p_1'' = 0$, results in the following equation

$$0 = \Big((1+\Sigma) - (1 + p_2' p_3') \Big) \; p_1' \tag{81}$$

and, since $p_1' \overset{!}{\neq} 0$, choose $p_2' p_3' = \Sigma$ to solve it.

Table 6 shows that now all cases of p_0'' or p_1'' being $= 0$ or $\neq 0$ are met.

	for $p_1'' = 0$	for $p_1'' \neq 0$
and $p_0'' = 0$	$a = c = 0$, then choose $p_2' p_3' = \Sigma$ $\Rightarrow (1+\Sigma) - (1 + p_2' p_3') = 0$	$a = 0$, then choose $p_2' p_3' \neq \Sigma$ and $p_2' p_3' \neq -1$ $\Rightarrow p_1' = - \frac{(1 + p_2' p_3') p_1''}{(1+\Sigma) - (1 + p_2' p_3')}$
and $p_0'' \neq 0$	$c = 0$, then choose $p_2' p_3' \neq \Sigma$ $\Rightarrow p_1' = \frac{(1+\Sigma) - (1 + p_2' p_3')}{p_0''}$	choose $p_2' p_3' = -1$ $\Rightarrow c = 0$ and $D = b^2$ $\Rightarrow$ quadratic equation solvable, $p_1' = -\frac{b}{a}$

Table 6: All cases of p_0'' or p_1'' being $= 0$ or $\neq 0$ are met.

Hence, one can solve this system for any combination of p_i'', therefore all points of the affine space are within at least the 'second' quadric for even dimension $d \geq 4$.

In Odd Dimensions

Now show the correlation in the the case of an *odd dimension* $d \geq 3$.
Choose the quadric to be of the form as in Eq. (20) above in Section 2.3

$$\hat{Q} = \begin{pmatrix} B & O & ... & O & O \\ O & B & ... & O & O \\ & & ... & & \\ O & O & ... & B & O \\ O & O & ... & O & \begin{matrix} 1 & 0 \\ 0 & -q^2 \end{matrix} \end{pmatrix}, \tag{82}$$

with q either a square or a non-square number, and B and O the same 2×2 matrices as above. The matrix is invertible ($\det(Q) = (-1)^{\frac{d-1}{2}}(-q^2) \neq 0$) and symmetric, therefore it is indeed a representation of a quadric. Also, it preserves the affine space.
With a matrix of this form one has the equation for the points $\hat{p}$ on the quadric as

$$\begin{aligned} \hat{p}^t \hat{Q} \hat{p} &= \sum_{i=0}^{d} \sum_{j=0}^{d} p_i \ \hat{Q}_{ij} \ p_j \\ &= 2 \ p_0 p_1 + 2 \ p_2 p_3 + ... + 2 \ p_{d-3} p_{d-2} + p_{d-1}^2 - \ q^2 \\ &= 2 \ p_0 p_1 + \sum_{i=1}^{\frac{d-3}{2}} 2 \ p_{2i} \ p_{2i+1} + p_{d-1}^2 - \ q^2 \\ &\stackrel{!}{=} 0 \end{aligned} \tag{83}$$

Again, taking this equation as the restriction for the p_0 component and having arbitrary parameters p_i for the other components, one has the following form for the quadric points

$$\hat{p} = \begin{pmatrix} -\frac{\sum_{i=1}^{\frac{d-3}{2}} p_{2i} \ p_{2i+1} + p_{d-1}^2 - q^2}{p_1} \\ p_1 \\ p_2 \\ ... \\ p_{d-1} \\ 1 \end{pmatrix}, \tag{84}$$

with $p_i \in \mathbb{F}$ for $i \geq 2$ and $p_1 \in \mathbb{F} \backslash \{0\}$. So, the only parameter restriction is $p_1 \neq 0$.
Doing the same steps regarding the translation by a quadric point, one has the following equation for a point on the 'second' quadric.

$$\hat{p}'' = \begin{pmatrix} -\frac{\sum_{i=1}^{\frac{d-3}{2}} p_{2i} \ p_{2i+1} + p_{d-1}^2 - q^2}{p_1} - \frac{\sum_{i=1}^{\frac{d-3}{2}} p'_{2i} \ p'_{2i+1} + p'^2_{d-1} - q^2}{p'_1} \\ p_1 + p'_1 \\ p_2 + p'_2 \\ ... \\ p_{d-1} + p'_{d-1} \\ 1 \end{pmatrix} \tag{85}$$

Now choosing, w.l.o.g., $p'_i = 0$ for $2 \leq i \leq d-2$ (here, do not choose $p'_{d-1} = 0$), defining $\sum_{i=1}^{\frac{d-3}{2}} p_{2i}\, p_{2i+1} =: \Sigma'$ and doing the same steps as above results in

$$\begin{aligned}
p''_0 &= -\frac{\Sigma' + p_{d-1}^2 - q^2}{p''_1 - p'_1} - \frac{p'^2_{d-1} - q^2}{p'_1} \\
\Leftrightarrow\ 0 &= p''_0\big(p''_1 - p'_1\big)p'_1 + \big(\Sigma' + p_{d-1}^2 - q^2\big)p'_1 + \big(p'^2_{d-1} - q^2\big)\big(p''_1 - p'_1\big) \\
0 &= \Big(-p''_0\Big)p'^2_1 + \Big(p''_0 p''_1 + \big(\Sigma' + p_{d-1}^2 - q^2\big) - \big(p'^2_{d-1} - q^2\big)\Big)p'_1 + \Big(\big(p'^2_{d-1} - q^2\big)\, p''_1\Big) \\
&= a\, p'^2_1 + b\, p'_1 + c\ . \qquad (86)
\end{aligned}$$

This equation now has again the form of a quadratic equation in p'_1. The discriminant $D = b^2 - 4ac$ is a square number, e.g., if $a = 0$ or $c = 0$.
First, consider the case $p''_0 \neq 0$ and $p''_1 \neq 0$. There the condition $c = 0$ can be met by choosing the free parameter p'_{d-1} to q.

$$\begin{aligned}
p'_{d-1} &= q \qquad (87) \\
\Leftrightarrow c &= \big(p'^2_{d-1} - q^2\big)\, p''_1 = \big(q^2 - q^2\big)\, p''_1 = 0
\end{aligned}$$

Only, since $p'_1 \overset{!}{\neq} 0$, one here also has the requirement $b \neq 0$, which is fulfilled.
From Eq. (87) it follows that

$$\begin{aligned}
p'_0 &= -\frac{q^2 - q^2}{p'_1} = 0 \\
\Rightarrow p''_0 &= -\frac{\Sigma' + p_{d-1}^2 - q^2}{p_1} + 0 \\
\Rightarrow \big(\Sigma' + p_{d-1}^2 - q^2\big) &= -p''_0 p_1 \qquad (88)
\end{aligned}$$

So,

$$\begin{aligned}
b &= p''_0 p''_1 + \big(\Sigma' + p_{d-1}^2 - q^2\big) - \big(p'^2_{d-1} - q^2\big) \\
&= p''_0 p''_1 - p''_0 p_1 - \big(q^2 - q^2\big) \\
&= p''_0 (p''_1 - p_1) \\
&= p''_0 p'_1 \neq 0 \qquad (89)
\end{aligned}$$

since $p''_0 \neq 0$ and $p'_1 \neq 0$.

In the special case $p''_0 = 0$, but $p''_1 \neq 0$, it follows that $a = 0$ and the quadratic equation in Eq. (86) becomes a linear one.

$$0 = \Big(\big(\Sigma' + p_{d-1}^2 - q^2\big) - \big(p'^2_{d-1} - q^2\big)\Big)p'_1 + \Big(\big(p'^2_{d-1} - q^2\big)\, p''_1\Big) \qquad (90)$$

This equation can be solved by

$$p'_1 = -\frac{\big(p'^2_{d-1} - q^2\big)\, p''_1}{\big(\Sigma' + p_{d-1}^2 - q^2\big) - \big(p'^2_{d-1} - q^2\big)}\ . \qquad (91)$$

This requires to choose $p'^2_{d-1} \neq \Sigma' + p_{d-1}^2$ (to not divide by zero) and $p'_{d-1} \neq q$ (since $p'_1 \overset{!}{\neq} 0$). However, it is still possible to choose p'_{d-1} such that it fulfills both.

In the opposite special case $p''_1 = 0$, but $p''_0 \neq 0$, it follows that $c = 0$ and the quadratic equation becomes also a linear one when dividing by p'_1.

$$0 = \Big(- p''_0 \Big) p'_1 + \Big(\big(\Sigma' + p_{d-1}^2 - q^2 \big) - \big(p'^2_{d-1} - q^2 \big) \Big) \tag{92}$$

This equation can be solved by

$$p'_1 = \frac{\big(\Sigma' + p_{d-1}^2 - q^2 \big) - \big(p'^2_{d-1} - q^2 \big)}{p''_0} \ . \tag{93}$$

This also requires to choose $p'^2_{d-1} \neq \Sigma' + p^2_{d-1}$ (since $p'_1 \overset{!}{\neq} 0$).

The last special case, where $p''_0 = 0$ and $p''_1 = 0$, results in the following equation

$$0 = \Big(\big(\Sigma' + p_{d-1}^2 - q^2 \big) - \big(p'^2_{d-1} - q^2 \big) \Big) p'_1 \tag{94}$$

and, since $p'_1 \overset{!}{\neq} 0$, choose $p'^2_{d-1} = \Sigma' + p^2_{d-1}$ to solve it.

Table 7 shows that now all cases of p''_0 or p''_1 being $= 0$ or $\neq 0$ are met.

	for $p''_1 = 0$	for $p''_1 \neq 0$
and $p''_0 = 0$	$a = c = 0$, then choose $p'^2_{d-1} = \Sigma' + p^2_{d-1}$ $\Rightarrow \big(\Sigma' + p^2_{d-1} - q^2 \big) - \big(p^2_{d-1} - q^2 \big) = 0$	$a = 0$, then choose $p'^2_{d-1} \neq \Sigma' + p^2_{d-1}$ and $p'_{d-1} \neq q$ $\Rightarrow p'_1 = - \frac{\left(p'^2_{d-1} - q^2 \right) p''_1}{\left(\Sigma' + p^2_{d-1} - q^2 \right) - \left(p'^2_{d-1} - q^2 \right)}$
and $p''_0 \neq 0$	$c = 0$, then choose $p'^2_{d-1} \neq \Sigma' + p^2_{d-1})$ $\Rightarrow p'_1 = \frac{\left(\Sigma' + p^2_{d-1} - q^2 \right) - \left(p'^2_{d-1} - q^2 \right)}{p''_0}$	choose $p'_{d-1} = q$ $\Rightarrow c = 0$ and $D = b^2$ $\Rightarrow$ quadratic equation solvable, $p'_1 = -\frac{b}{a}$

Table 7: All cases of p''_0 or p''_1 being $= 0$ or $\neq 0$ are met.

Hence, one can solve the system for any combination of p''_i, therefore all points of the affine space are within at least the 'second' quadric for any dimension - even or odd - $d \geq 3$.
Also it is not possible for the diameter to be one, since the number of points in only one quadric (or even biquadric) is too small to cover all p^d points of the affine space (which is also true in the $d = 2$ case).

Therefore, $diam = 2$ for $d \geq 3$. □

4.2.3 Graph Diameter in Dimension $d = 2$

Proposition. A graph generated by the translation of a quadric in the affine space has diameter $diam \leq 3$ for dimension $d = 2$.

Proof. Since this is in an *even dimension* the quadric is chosen to be of the form

$$\hat{Q} = \begin{pmatrix} B & \vec{0} \\ \vec{0}^t & 2 \end{pmatrix} \tag{95}$$

with B the same matrix as above.
The same construction of the points on the 'second' quadric lead to the following expression

$$\hat{p}'' = \begin{pmatrix} -\frac{1}{p_1} - \frac{1}{p_1'} \\ p_1 + p_1' \\ 1 \end{pmatrix} . \tag{96}$$

Since it is to prove that the diameter is $diam \leq 3$ and not only $diam = 2$ [23] one has to go one step further onto the 'third' quadric to reach every point in the affine space. So, one has, analogue to the translation steps above, the equation for a 'third' quadric point $\hat{p}''''$ as

$$\hat{p}'''' = \begin{pmatrix} -\frac{1}{p_1} - \frac{1}{p_1'} - \frac{1}{p_1'''} \\ p_1 + p_1' + p_1''' \\ 1 \end{pmatrix} \tag{97}$$

The only other parameter restriction (next to $p_1 \neq 0$ and $p_1' \neq 0$) is $p_1''' \neq 0$.
With this, one has the following system of equations

$$p_0'''' = -\frac{1}{p_1} - \frac{1}{p_1'} - \frac{1}{p_1'''} \tag{98}$$

$$\begin{aligned} p_1'''' &= p_1 + p_1' + p_1''' \\ &\Leftrightarrow p_1 = p_1'''' - p_1' - p_1''' \end{aligned} \tag{99}$$

Using this, one has

$$\begin{aligned} p_0'''' &= -\frac{1}{(p_1'''' - p_1' - p_1''')} - \frac{1}{p_1'} - \frac{1}{p_1'''} \\ \Leftrightarrow \;\; 0 &= p_0''''(p_1'''' - p_1' - p_1''')p_1'p_1''' + p_1'p_1''' + (p_1'''' - p_1' - p_1''')p_1''' + (p_1'''' - p_1' - p_1''')p_1' \\ 0 &= \Big(-1 - p_0''''p_1' \Big) \; p_1'''^2 + \Big(p_0''''(p_1'''' - p_1')p_1' + p_1' - p_1' + (p_1'''' - p_1') \Big) \; p_1''' + \Big((p_1'''' - p_1')\, p_1' \Big) \\ &= a \; p_1'''^2 + b \; p_1''' + c \; . \end{aligned} \tag{100}$$

This quadratic equation in p_1''' is solvable if the discriminant $D = b^2 - 4ac$ is a square number. This is fulfilled, e.g., if $a = 0$ or $c = 0$. Both conditions can be met here since p_1' still is a free parameter (only restricted by $p_1' \neq 0$).

$$\begin{aligned} a = 0 &= -1 - p_0''''p_1' \\ \Leftrightarrow \;\; p_1' &= -\frac{1}{p_0''''} \end{aligned} \tag{101}$$

$$\begin{aligned} c = 0 &= \left(p_1'''' - p_1'\right) p_1' \\ \Leftrightarrow \;\; p_1' &= p_1'''' \end{aligned} \tag{102}$$

[23] Indeed, examples were found of graphs with $diam = 3$ in $d = 2$, so $diam = 2$ can not be true in general. Also, examples with $diam = 2$ were found, so $diam = 3$ can not be an equality, either.

in the first case one has the restriction $p_0'''' \neq 0$ (to not divide by zero) and in the second case $p_1'''' \neq 0$ (since $p_1' \overset{!}{\neq} 0$).
For the case $p_0'''' = p_1'''' = 0$ the point in question is the center $\hat{c} = \langle\, 0, 0, 1\,\rangle^t$ of the first quadric, which lies on every second quadric, since for translated quadrics any point (e.g., the center of one quadric) is a neighbour (quadric point) of all its neighbours (quadric points).
Hence, one can solve this system for any combination of p_i'''', therefore all points of the affine space are within at least the 'third' quadric for the dimension $d \geq 2$. □

4.2.4 Comment

The observant reader might have noticed that the prove given above does not build on the fact that the field $\mathbb{F}$ one performs these calculations in is a finite one, the only restriction is in the characteristics of the quadrics that are considered. Therefore, it is also true in other fields, e.g., the real numbers $\mathbb{R}$.
In the real numbers the solutions to the quadric equations yield conics, i.e., ellipses, parabolas and hyperbolas.

View the setup given above in the projective space with a center chosen such that it has the 2-dimensional real space $\mathbb{R}^2$ as corresponding affine space.
There, the quadric equation $\hat{p}^t\hat{Q}\hat{p} = 0$ with the quadric given above in the 2-dimensional case

$$\hat{Q} = \begin{pmatrix} B & \vec{0} \\ \vec{0}^t & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{103}$$

has the form

$$\hat{p}^t\hat{Q}\hat{p} = \left\langle \begin{smallmatrix} x \\ y \\ 1 \end{smallmatrix} \right\rangle^t \hat{Q} \left\langle \begin{smallmatrix} x \\ y \\ 1 \end{smallmatrix} \right\rangle = 2xy + 2 = 0 \tag{104}$$

$$\Leftrightarrow y = -\frac{1}{x}, \tag{105}$$

which is the form of a *hyperbola*.
And indeed, if one looks at translated hyperbolas in the 2-dimensional real space it is easy to imagine that the whole space can be reached by three steps. In Figures 14 and 15 one can see plots of once and twice translated hyperbolas, respectively. In the case of considering only once translated hyperbolas it is easily imaginable that the region around the part of the original hyperbola (the one centered at the origin) that is closest to the center can not be reached by a translate hyperbola with center somewhere on the original one. When going three steps from the center, however, also this part can be reached and the whole 2-dimensional plane (up to infinity through the hyperbola arms going to infinity) can be reached.
It is also nice to see that the translated hyperbolas go through the center of the original hyperbola their center is on, i.e., $\hat{p}_1 \in Q_{\hat{p}_2} \Leftrightarrow \hat{p}_2 \in Q_{\hat{p}_1}$, which shows the mutual neighbouring relationship for translated quadrics.

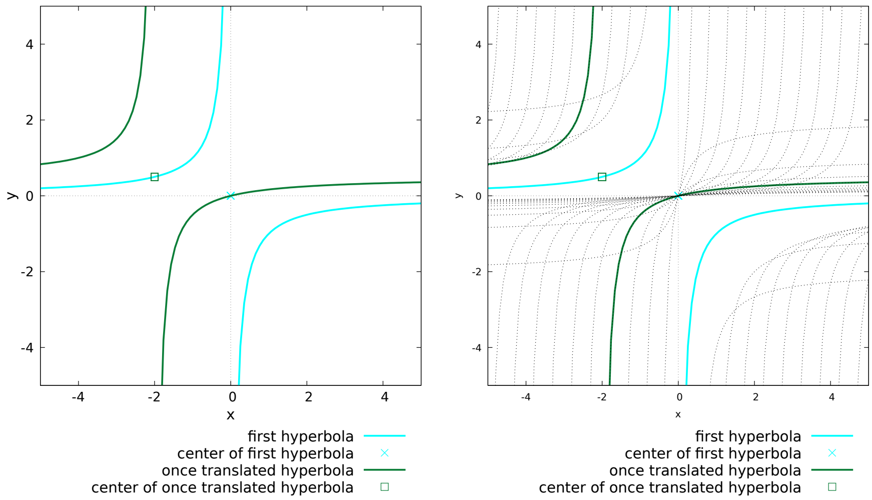


Figure 14: *left*: Original hyperbola centered at the origin and one translated hyperbola with its origin on the original hyperbola. *right*: The same original hyperbola centered at the origin and some hyperbolas from the family of curves that have their origin at one point of the original hyperbola.

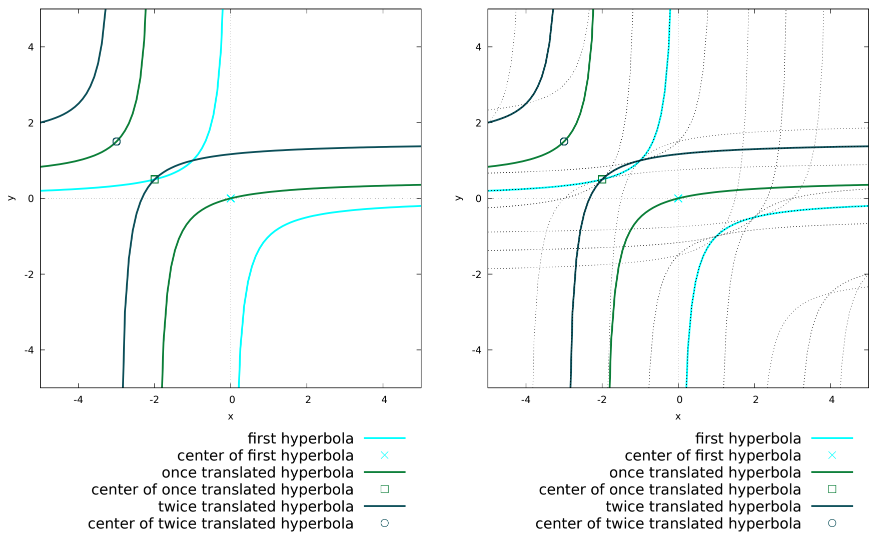


Figure 15: *left*: Original hyperbola centered at the origin, one translated hyperbola with its origin on the original hyperbola, and one twice translated hyperbola with its origin on the first translated hyperbola. *right*: The same original hyperbola centered at the origin and some hyperbolas from the family of curves that have their origin at one point of one of the once translated hyperbolas, i.e., both translation parameters for the first and the second step are changed for these.

4.3 Diameter of the Graph of an Arbitrary Biquadric Field

Now, consider arbitrary quadric fields instead of ones consisting of translated quadrics. These are fields where any quadric (or conic) can be attached to the different points, just restricted by the available characteristics for each dimension and the requirement that they should preserve the affine space, i.e., the plane at infinity.
This translates into the restriction that $\mathbf{G}_d$ can be chosen arbitrarily, as long as it still is symmetric and invertible.
Due to projective transformations it is, w.l.o.g., possible to choose the first quadric as the same one as above given in Eqs. (64) and (82). The second one then can have arbitrary components within $\mathbf{G}_d$ (as long as it still is symmetric and invertible).
For example in dimension $d = 4$ the second quadric Q_2 has the general quadric form

$$\hat{Q}_2 = \begin{pmatrix} a & b & c & d & 0 \\ b & e & f & g & 0 \\ c & f & h & i & 0 \\ d & g & i & j & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{106}$$

with 10 arbitrary parameters a to j, only restricted by $\det(\mathbf{G}_4) \neq 0$.
Translating this quadric to a point within the 'first' quadric and calculating its quadric points (the 'second' quadric points), then calculating the form of one component of a 'second' quadric point results in a formula depending on the 10 parameters of the quadric Q_2, the coordinates describing the 'first' quadric point that is chosen to be the center of the translated quadric (restricted by the quadric equation for the first quadric) and the other coordinates of this point.
The number of parameters increases when increasing the dimension.
Also, since the relation $\hat{p}_1 \in Q^{\pm}_{\hat{p}_2} \Leftrightarrow \hat{p}_2 \in Q^{\pm}_{\hat{p}_1}$ does not hold for arbitrary translated quadrics, the structure of the generated graph doesn't look the same from any point any more.
For these reasons, a proof of a similar statement as above is not done here.

However, some further considerations can be done when looking at the analogy to the conics in the real space. The conics that occur there in the non-translated cases still have to be hyperbolas (these correspond to the one characteristic that is allowed in finite projective geometries with even dimension), but they can be scaled arbitrarily. It might be possible to construct a quadric field such that all hyperbolas two steps away from the center (their origin is on a hyperbola whose origin is on the original hyperbola) still leave a part of the plane uncovered. However, such a quadric field is probably quite rare.

When going to finite fields the occurrence of such a special field is probably similar. Also here, by careful construction, it might be possible to find a quadric field that doesn't allow reaching the whole field by two or three steps.
But these exceptions of larger diameter fields are probably very rare, especially when going to *biquadric fields* where all the points do have a neighbour in any direction.
Also, some calculations of the diameter of a graph generated from a 3-dimensional finite projective geometry equipped with a biquadric field whose quadric forms were allocated randomly were done. For all examples investigated the diameter turned out to be $diam = 2$.

From these considerations we can conjecture that most (but maybe not all) of the biquadric fields do have a very small graph diameter of 2 or 3, as in the case of the translated quadrics.

4.4 Summary

For the graph generated by quadric fields built by the translation of one quadric it was shown that the diameter $diam$ of the graph is quite small by being $diam = 2$ in the general case with dimension $d \geq 3$ and $diam \leq 3$ in dimension $d = 2$.

In the general case of a non-translated quadric field the proof of such a relation is (at least) very difficult.
However, good reasons are given to conjecture that most (but maybe not all) of the biquadric fields have a very small graph diameter, as in the case of the translated quadrics.

5 Overview and Outlook

This work presented the following achievements in the considerations on a finite projective geometry as a quantum world:

First, in Section 3 the *local world domain*, defining a subspace in the geometry where an Euclidean-like ordering of the points is possible, and the transformations between different local world domains were investigated.
The prior assumption was that biquadric points different from the ones selected as new basis vectors for the local coordinate system are not part of the local world domain. It is proven by a counter example that this assumption has to be dropped since it is indeed possible that these can be within the local world domain.
Moreover, for the transformation statistics investigating the possible transformations from the local world domain of one local coordinate system into the local world domain of other local coordinate systems some common traits for all considered cases are found. Most of the points within a local world domain can be successfully transformed into other local world domains about $2p$ times, with a few ($\lesssim 3\%$) about twice as much. The shape of the distribution resembles a Gaussian distribution and gets narrower for increasing prime numbers.

Furthermore, in Section 4, the finite field and the neighbouring relations which are defined by the biquadrics were interpreted as a *graph* and its diameter was explored.
It is proven that the diameter of a quadric field is $diam = 2$ in the general case of $d \geq 3$ and $diam \leq 3$ for $d = 2$.
For non-translated biquadric fields good reasons are given to conjecture that their diameter is also very small in most of the cases.

This small diameter leads to a very small world when describing spacetime using a full biquadric field. Such a small world, however, is not well-suited to describe a world of the size of the visible universe. The idea for ongoing research is to replace the full biquadric field with a thinner one, i.e., replacing points in the geometry equipped with a full biquadric field by points that have no biquadric attached. The 'dead ends' for a walk through the graph appearing during this process will increase the diameter of the describing graph. One has to be careful during this process that the graph is still connected at the end to avoid spacetime regions that can not be reached.
Also defining a 'forwards' and a 'backwards' direction on a selected time-like line through the center, i.e., defining the 'future' and the 'past' of the center as the biquadric points in the 'forwards' and 'backwards' directions of the time-like line, and allowing only walks through the graph that go into the future will influence the diameter.

Ongoing from these results further investigations can be done.

- ◇ One can further investigate the normalization of the basis vectors for the local coordinate systems to motivate a conceptual decision regarding which basis vectors have to be normalized.
- ◇ Also, one can investigate the local world domain and the transformations between local coordinate systems in higher dimensions than $d = 2$.
- ◇ One can go on searching for an equation or an upper bound for the diameter of the non-translated quadric field.

- ◇ One can investigate how thin the not-full biquadric field has to get in order for the diameter to rise to a size better suited to describe the visible universe.
- ◇ And still the final goal of the finite projective geometry as a quantum world is to get to a continuum limit going to very high prime number in the order of $p \sim 10^{123}$ (or even larger) to see if the model is suited to describe our (perhaps only continuous-looking) real world. The order $p \sim 10^{123}$ is calculated by dividing the size of the visible universe by the Planck length ℓ_P. This fraction gives the size of the local world domain of the finite geometry and has to be squared to give the full prime of the field.

  In the best case, all measured effects and quantities match the predictions made by this theory. Furthermore, the hope is that it is possible to find other measurable but yet unmeasured quantities that are predicted differently in the scope of the finite projective geometry than in other models of quantum worlds to test the theories against each other.

In conclusion, this thesis gave some interesting results regarding the finite projective geometry as an attempt to a unification of quantum field theory and general relativity. Some further research can grow from these results and I wish the best and much success to my associate researchers and the project.

References

[Beutelspacher and Rosenbaum, 2004] Beutelspacher, A. and Rosenbaum, U. (2004). *Projektive Geometrie (german)*. Friedrich Vieweg & Sohn Verlag, second edition.

[Bondy and Murty, 1976] Bondy, J. and Murty, U. (1976). *Graph Theory with Applications*. The MacMillan Press LTD, London and Basingstoke, first edition.

[Estrada and Knight, 2015] Estrada, E. and Knight, P. A. (2015). *A First Course in Network Theory*. Oxford University Press, first edition.

[Gauss, 1986] Gauss, C. F. (1986). *Disquisitiones Arithemeticae (translator into English: Clarke, Arthur A.)*. Springer, second, corrected edition.

[Gimperlein, 2018] Gimperlein, M. (2018). Biquadric fields and game of life in finite geometries (master's thesis) (and private communication).

[Hirschfeld and Thas, 2016] Hirschfeld, J. W. P. and Thas, J. A. (2016). *General Galois Geometries*. Springer, first edition.

[Laska, 2018] Laska, A. (2018). private communication.

[Lidl and Niederreiter, 1994] Lidl, R. and Niederreiter, H. (1994). *Introduction to Finite Fields and their Applications*. Cambridge University Press, second edition.

[Mecke, 2017] Mecke, K. (2017). Biquadrics configure finite projective geometry into a quantum spacetime. *EPL (Europhysics Letters)*, 120(1).

[Mecke, 2018] Mecke, K. (2018). private communication.

[Moorhouse, 2007] Moorhouse, E. G. (2007). Incidence geometry. `http://www.math.ucr.edu/home/baez/qg-fall2016/incidence_geometry.pdf` (accessed in December 2018).

[Rahman, 2017] Rahman, M. S. (2017). *Basic Graph Theory*. Springer, first edition.

[Richter-Gebert, 2001] Richter-Gebert, J. (2001). *Perspectives on Projective Geometry*. Springer-Verlag, first edition.

[Richter-Gebert and Orendt, 2009] Richter-Gebert, J. and Orendt, T. (2009). *Geometriekalkül (german)*. Springer-Verlag, first edition.

A To the Calculations Transforming the Local World Domain

A.1 Parameters of the Histogram Analysis for High Prime Numbers in the $\mathbf{G}_2 = \mathbf{G}_2^{101}$ Case

These are the parameters that are used for the calculations done for the peak analysis in Section 3.4.5.

$$
\begin{aligned}
p &= 1999 \\
&\quad f = 7 \\
&\quad \hat{q}^0 = \left\langle \begin{smallmatrix} 4 \\ 961 \\ 1 \end{smallmatrix} \right\rangle \in Q^+ \cap \mathfrak{A}^2 \qquad \text{and} \qquad \hat{q}^1 = \left\langle \begin{smallmatrix} 0 \\ 681 \\ 1 \end{smallmatrix} \right\rangle \in Q^- \cap \mathfrak{A}^2 \\
p &= 2999 \\
&\quad f = 23 \\
&\quad \hat{q}^0 = \left\langle \begin{smallmatrix} 4 \\ 1304 \\ 1 \end{smallmatrix} \right\rangle \in Q^+ \cap \mathfrak{A}^2 \qquad \text{and} \qquad \hat{q}^1 = \left\langle \begin{smallmatrix} 0 \\ 219 \\ 1 \end{smallmatrix} \right\rangle \in Q^- \cap \mathfrak{A}^2 \\
p &= 4003 \\
&\quad f = 5 \\
&\quad \hat{q}^0 = \left\langle \begin{smallmatrix} 1 \\ 514 \\ 1 \end{smallmatrix} \right\rangle \in Q^+ \cap \mathfrak{A}^2 \qquad \text{and} \qquad \hat{q}^1 = \left\langle \begin{smallmatrix} 0 \\ 1879 \\ 1 \end{smallmatrix} \right\rangle \in Q^- \cap \mathfrak{A}^2 \\
p &= 5003 \\
&\quad f = 6 \\
&\quad \hat{q}^0 = \left\langle \begin{smallmatrix} 1 \\ 1813 \\ 1 \end{smallmatrix} \right\rangle \in Q^+ \cap \mathfrak{A}^2 \qquad \text{and} \qquad \hat{q}^1 = \left\langle \begin{smallmatrix} 0 \\ 100 \\ 1 \end{smallmatrix} \right\rangle \in Q^- \cap \mathfrak{A}^2 \\
p &= 6007 \\
&\quad f = 6 \\
&\quad \hat{q}^0 = \left\langle \begin{smallmatrix} 2 \\ 2084 \\ 1 \end{smallmatrix} \right\rangle \in Q^+ \cap \mathfrak{A}^2 \qquad \text{and} \qquad \hat{q}^1 = \left\langle \begin{smallmatrix} 0 \\ 2901 \\ 1 \end{smallmatrix} \right\rangle \in Q^- \cap \mathfrak{A}^2 \\
p &= 7019 \\
&\quad f = 8 \\
&\quad \hat{q}^0 = \left\langle \begin{smallmatrix} 1 \\ 1791 \\ 1 \end{smallmatrix} \right\rangle \in Q^+ \cap \mathfrak{A}^2 \qquad \text{and} \qquad \hat{q}^1 = \left\langle \begin{smallmatrix} 0 \\ 3437 \\ 1 \end{smallmatrix} \right\rangle \in Q^- \cap \mathfrak{A}^2
\end{aligned}
$$

A.2 Position, Size and Shape Analysis for the $\mathbf{G}_2 = \mathbf{G}_2^{-101}$ Case

The results for the other biquadric with $\mathbf{G}_2 = \mathbf{G}_2^{-101}$ are shown here, since they show analogous behavior to the results presented above in Section 3.4.5 for the $\mathbf{G}_2 = \mathbf{G}_2^{101}$ case and are therefore not discussed in detail there.
These are the parameters that are used for the calculations done for the peak analysis analogous to the analysis done in Section 3.4.5 but with $\mathbf{G}_2 = \mathbf{G}_2^{-101}$ as

$$\mathbf{G}_2^{-101} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{107}$$

$$\begin{aligned}
&p = 1999 \\
&\qquad f = 7 \\
&\qquad \hat{q}^0 = \left\langle \begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix} \right\rangle \in Q^+ \cap \mathfrak{A}^2 \qquad \text{and} \qquad \hat{q}^1 = \left\langle \begin{smallmatrix} 0 \\ 681 \\ 1 \end{smallmatrix} \right\rangle \in Q^- \cap \mathfrak{A}^2 \\
&p = 2999 \\
&\qquad f = 23 \\
&\qquad \hat{q}^0 = \left\langle \begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix} \right\rangle \in Q^+ \cap \mathfrak{A}^2 \qquad \text{and} \qquad \hat{q}^1 = \left\langle \begin{smallmatrix} 0 \\ 219 \\ 1 \end{smallmatrix} \right\rangle \in Q^- \cap \mathfrak{A}^2 \\
&p = 4003 \\
&\qquad f = 5 \\
&\qquad \hat{q}^0 = \left\langle \begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix} \right\rangle \in Q^+ \cap \mathfrak{A}^2 \qquad \text{and} \qquad \hat{q}^1 = \left\langle \begin{smallmatrix} 0 \\ 1879 \\ 1 \end{smallmatrix} \right\rangle \in Q^- \cap \mathfrak{A}^2 \\
&p = 5003 \\
&\qquad f = 6 \\
&\qquad \hat{q}^0 = \left\langle \begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix} \right\rangle \in Q^+ \cap \mathfrak{A}^2 \qquad \text{and} \qquad \hat{q}^1 = \left\langle \begin{smallmatrix} 0 \\ 100 \\ 1 \end{smallmatrix} \right\rangle \in Q^- \cap \mathfrak{A}^2 \\
&p = 6007 \\
&\qquad f = 6 \\
&\qquad \hat{q}^0 = \left\langle \begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix} \right\rangle \in Q^+ \cap \mathfrak{A}^2 \qquad \text{and} \qquad \hat{q}^1 = \left\langle \begin{smallmatrix} 0 \\ 2901 \\ 1 \end{smallmatrix} \right\rangle \in Q^- \cap \mathfrak{A}^2
\end{aligned} \tag{108}$$

The calculation results are shown analogous to Section 3.4.5 above. Table 8 shows the position values, Table 9 the size values, Table 10 the fit parameters of the standard deviation in the Gaussian fit and Figure 16 the plots showing the shape of the first peak for the case where $\mathbf{G}_2 = \mathbf{G}_2^{-101}$. The heatmaps produced in these calculations for $p \in \{1999, 2999, 4003, 5003, 6007\}$ are shown in Section A.3.2.

p	$\frac{\mu}{p}$ of the first peak	$\frac{\mu}{p}$ of the second peak
1999	1.93	3.94
2999	1.93	3.92
4003	1.94	3.94
5003	2.00 (1.99972)	4.00 (3.99946)
6007	1.94	3.94

Table 8: Mean value of the peak μ divided by p for different primes p. The parameters for the calculations are given above in Eq. (108) with $\mathbf{G}_2 = \mathbf{G}_2^{-101}$.

prime p	number of points in the second peak	number of all points	percentage of the points in the second peak
1999	148	3968	3.73 %
2999	136	5928	2.29 %
4003	228	7920	2.88 %
5003	272	10200	2.67 %
6007	307	11880	2.56 %

Table 9: Number of points in the second peak, number of all points and percentage of the points in the second peak for different primes p. The parameters for the calculations are given above in Eq. (108) with $\mathbf{G}_2 = \mathbf{G}_2^{-101}$.

prime p	$a = \frac{\sigma}{\sqrt{p}}$
1999	2.349 ± 0.189
2999	2.152 ± 0.172
4003	2.303 ± 0.142
5003	2.220 ± 0.116
6007	2.142 ± 0.107

Table 10: Fit parameter a of the Gaussian distribution as given in Eq. (53) with mean μ and standard deviation $\sigma = a \cdot \sqrt{p}$ for different primes p. The parameters for the calculations are given above in Eq. (108) with $\mathbf{G}_2 = \mathbf{G}_2^{-101}$.

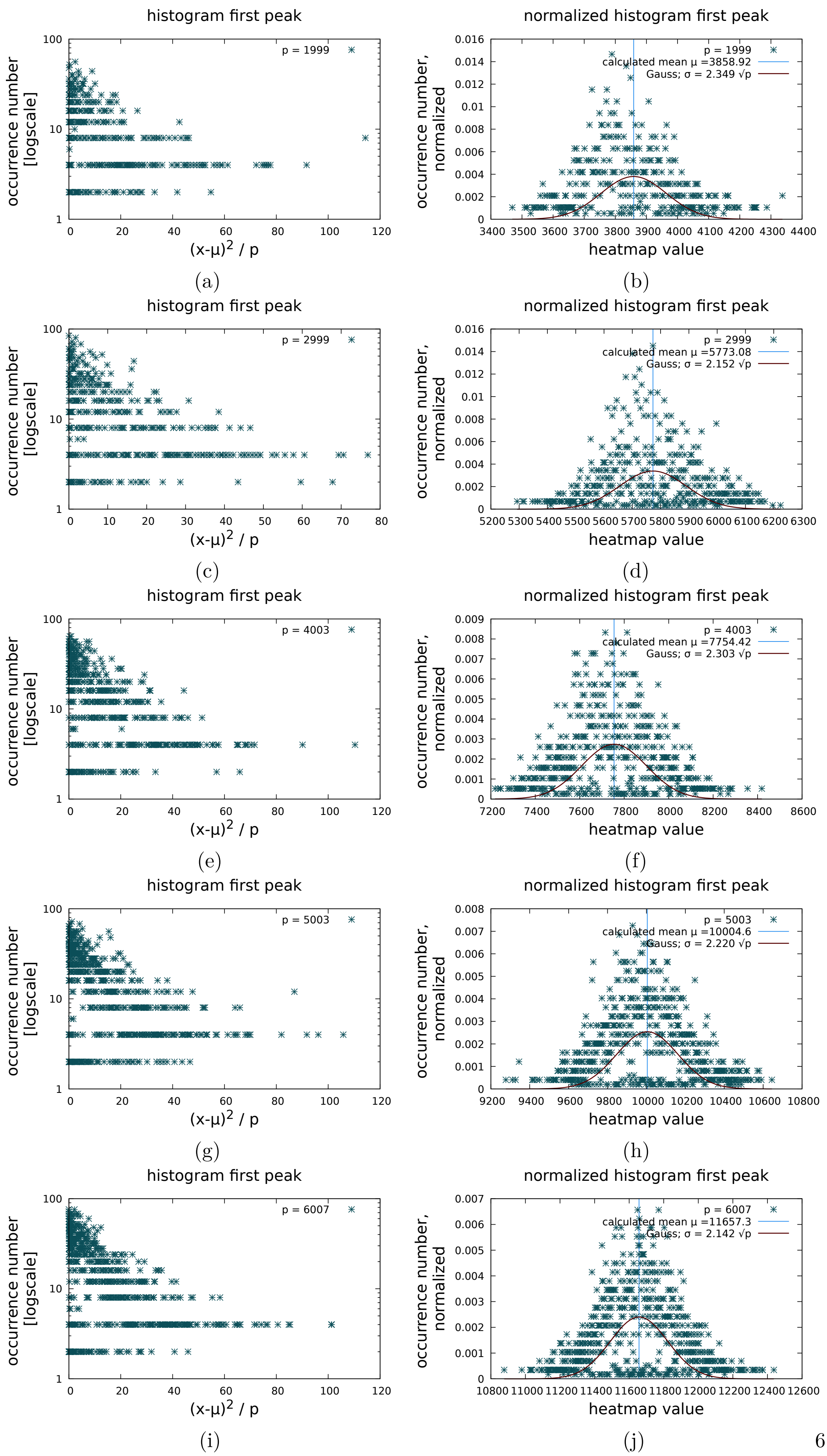


Figure 16: The first peak of the histogram for different primes p. The parameters for the calculations are given above in Eq. (108) with $\mathbf{G}_2 = \mathbf{G}_2^{-101}$.
left: The normalized peak, with the calculated mean ν and a Gaussian distribution as given in Eq. (53) fitted to it. *right:* The x axis is $\frac{(x-\mu)^2}{p}$ and the y axis is in logarithmic scale.

A.3 Heatmaps for Large Prime Numbers p

Here, the heatmaps for the analysis for large prime numbers $p \in \{1999, 2999, 4003, 5003, 6007\}$ are shown, both for the $\mathbf{G}_2 = \mathbf{G}_2^{101}$ and the $\mathbf{G}_2 = \mathbf{G}_2^{-101}$ case.

A.3.1 Heatmaps for the $\mathbf{G}_2 = \mathbf{G}_2^{101}$ Case

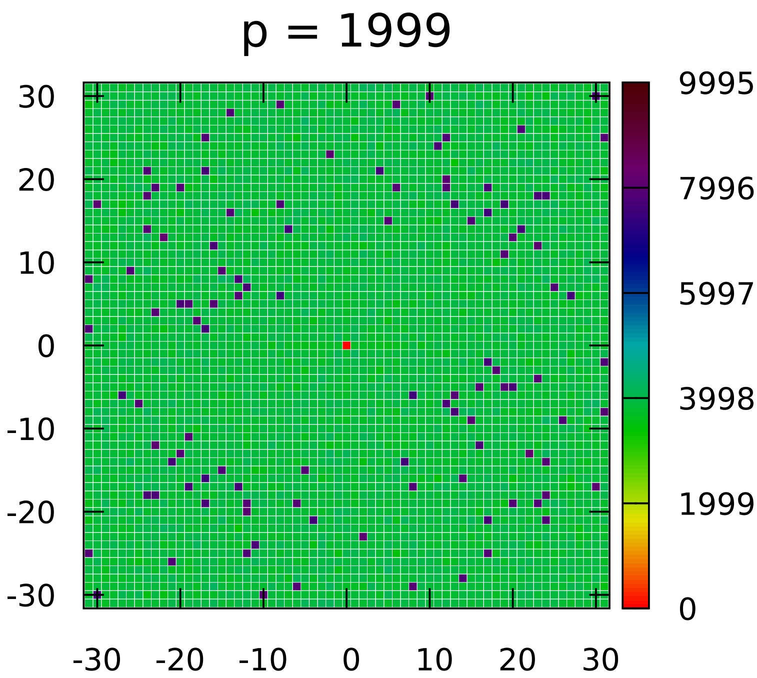
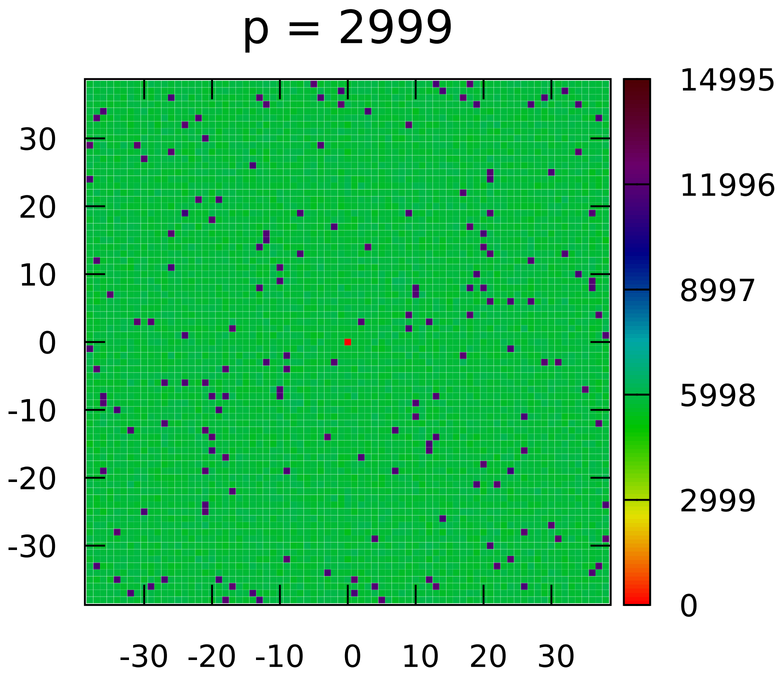


Figure 17: Heatmap of the selected local world domain for the prime number $p = 1999$. The parameters for the calculations are given in Appendix A.1 with $\mathbf{G}_2 = \mathbf{G}_2^{101}$.



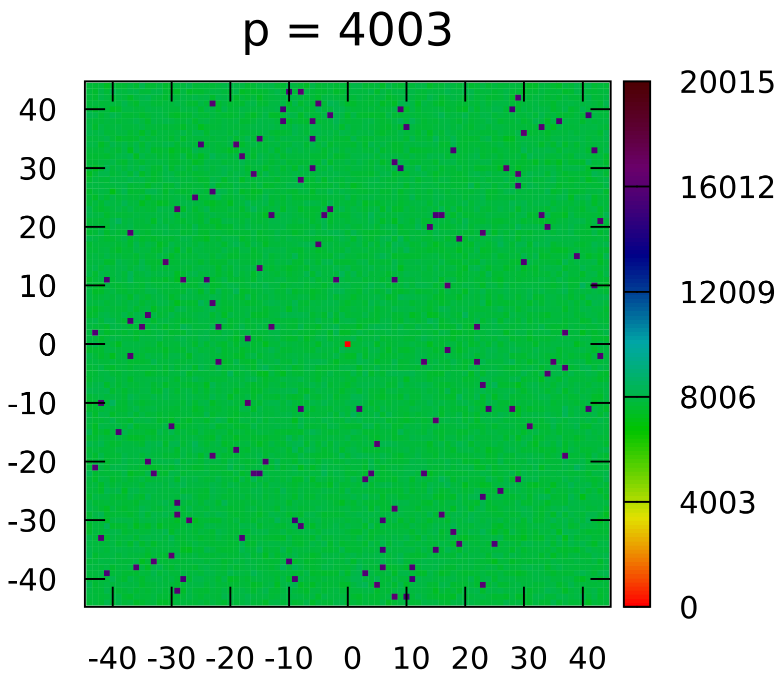
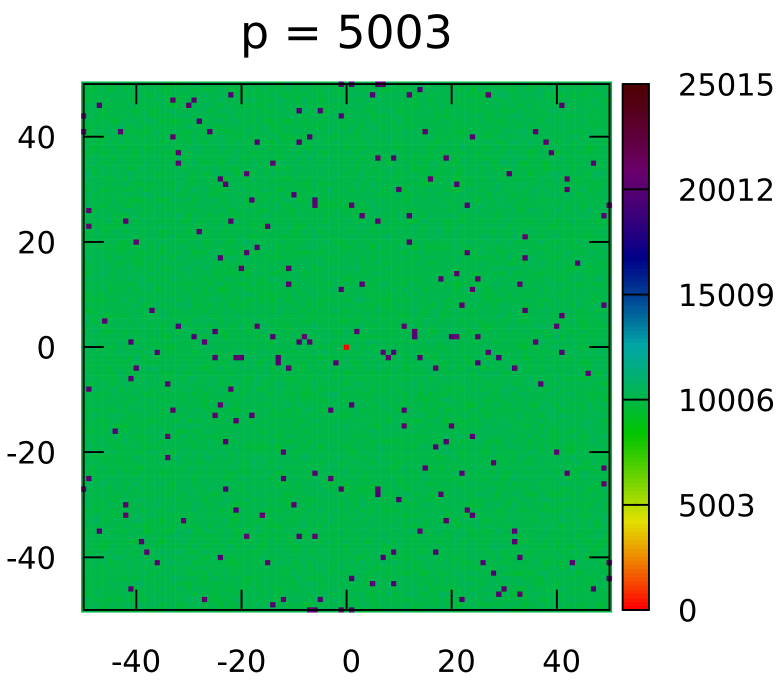


Figure 18: Heatmaps of the selected local world domains for the prime numbers $p \in \{2999\ (top), 4003\ (bottom)\}$. The parameters for the calculations are given in Appendix A.1 with $\mathbf{G}_2 = \mathbf{G}_2^{101}$.



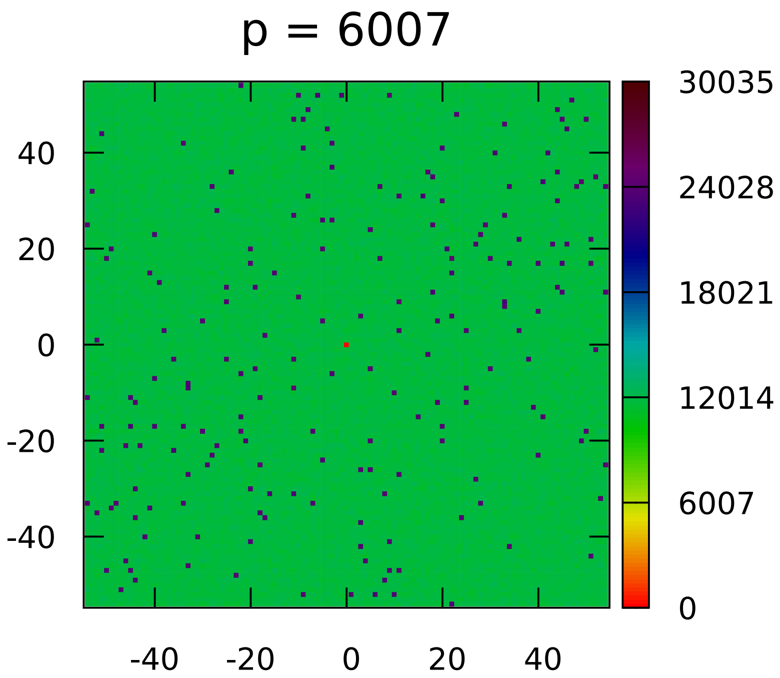


Figure 19: Heatmaps of the selected local world domains for the prime numbers $p \in \{5003\ (top), 6007\ (bottom)\}$. The parameters for the calculations are given in Appendix A.1 with $\mathbf{G}_2 = \mathbf{G}_2^{101}$.

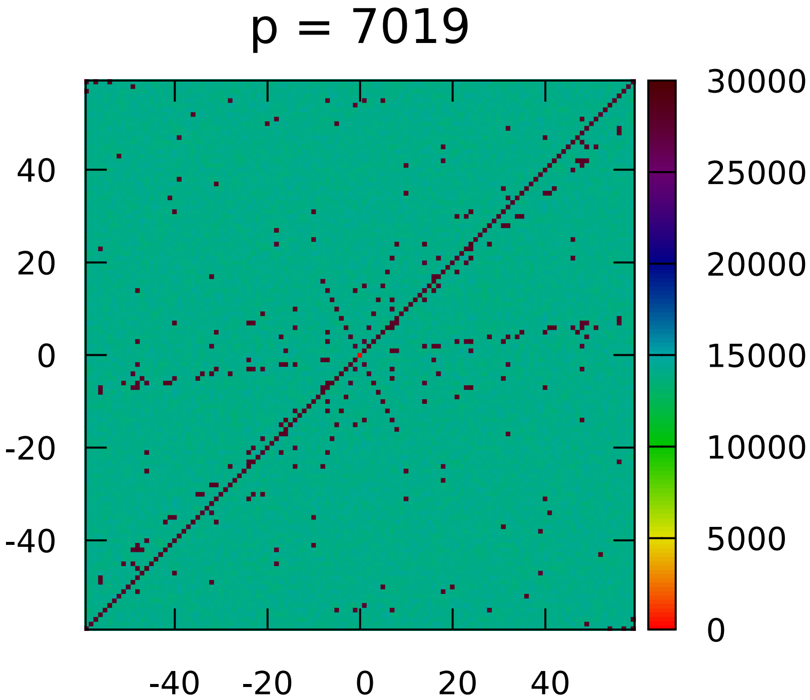


Figure 20: Heatmap of the selected local world domain for the prime number $p = 7019$. The parameters for the calculations are given in Appendix A.1 with $\mathbf{G}_2 = \mathbf{G}_2^{101}$.

A.3.2 Heatmaps for the $\mathbf{G}_2 = \mathbf{G}_2^{-101}$ Case

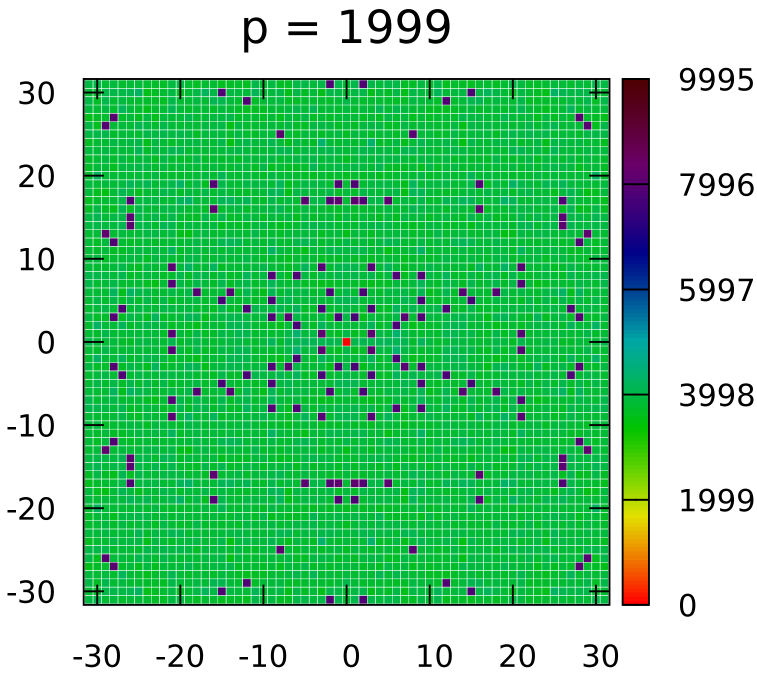
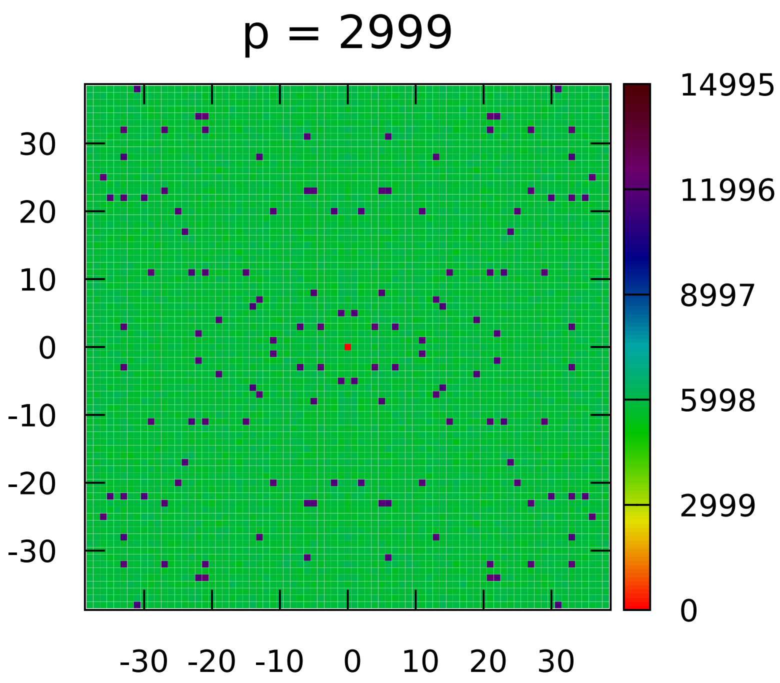


Figure 21: Heatmap of the selected local world domain for the prime number $p = 1999$. The parameters for the calculations are given above in Eq. (108) with $\mathbf{G}_2 = \mathbf{G}_2^{-101}$.



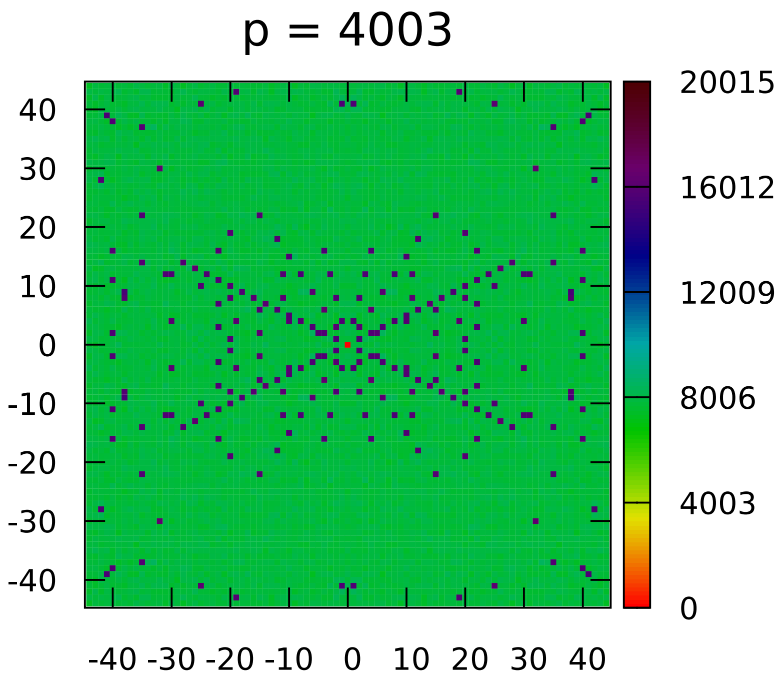
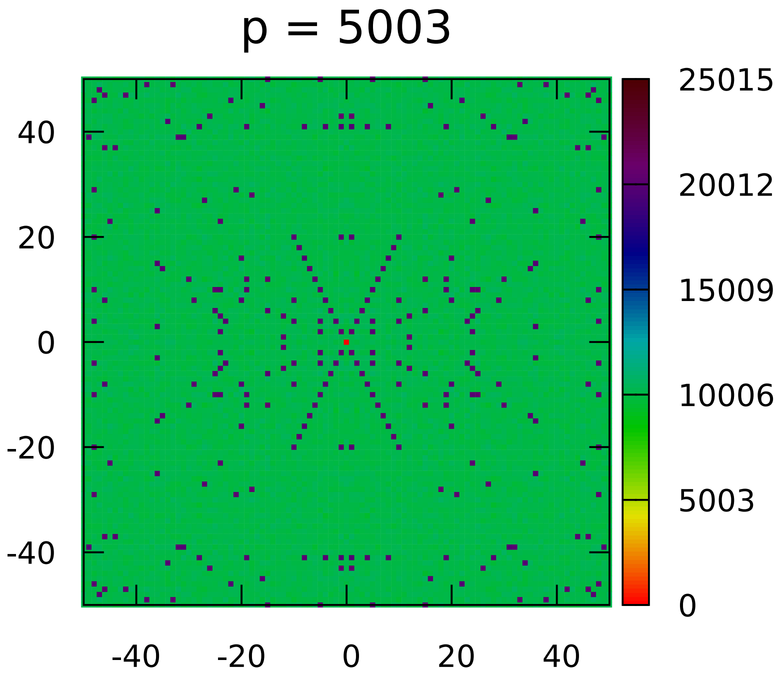


Figure 22: Heatmaps of the selected local world domains for the prime numbers $p \in \{2999\ (\mathit{top}), 4003\ (\mathit{bottom})\}$. The parameters for the calculations are given above in Eq. (108) with $\mathbf{G}_2 = \mathbf{G}_2^{-101}$.



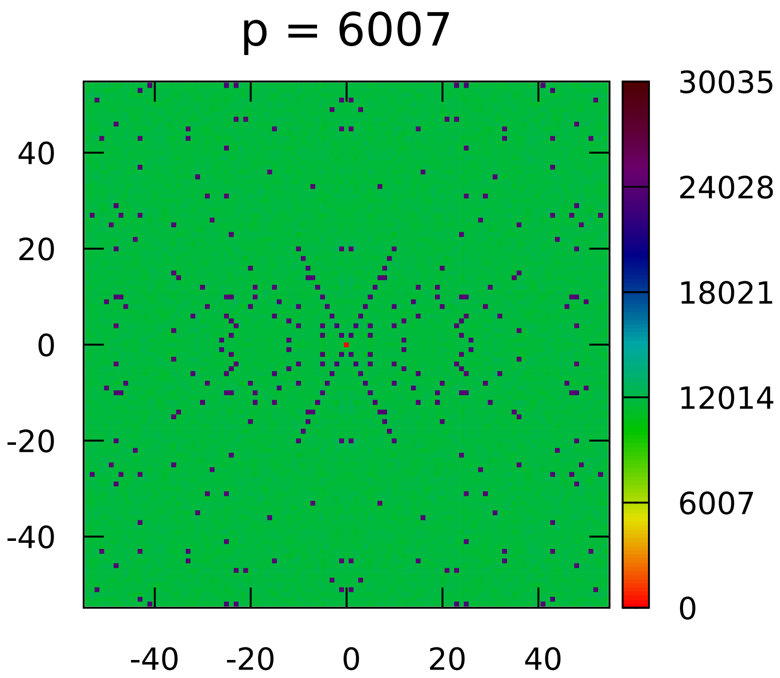


Figure 23: Heatmaps of the selected local world domains for the prime numbers $p \in \{5003\ (top), 6007\ (bottom)\}$. The parameters for the calculations are given above in Eq. (108) with $\mathbf{G}_2 = \mathbf{G}_2^{-101}$.

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Thank you very much! :)

Statutory declaration

I hereby confirm that I have written the accompanying master's thesis by myself, without contributions from any sources other than those cited in the text. This thesis was not submitted to any other authority to achieve an academic grading and was not published elsewhere.

Place, Date

Signature