Biquadric Fields and Game of Life in Finite Geometries

Master's Thesis in Physics

presented by

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December 20, 2018

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Abstract

Finite projective geometry is used to construct a new approach to the unification of space, time and matter, a new model of quantum gravity. In finite projective geometries $\mathbb{P}^d\mathbb{F}_q$[1] additional structure is needed to define neighborhood and distance - therefore the so-called *biquadric* $\mathcal{B}_{p_c}$ centered at the point p_c is introduced. The intersection points of the lines through p_c and the biquadric are called the *neighbors*. One equips every point in the projective space with a biquadric, defining a *biquadric field* $\mathfrak{B}$.

Squaring points in finite geometries leads to unintuitive results, as the periodicity of the finite field plays a role. Squaring objects is necessary when second order geometrical objects as biquadrics are introduced[2] - therefore a *local world domain* where the points do not feel the periodicity is defined. The second chapter investigates the refolding of lines into this quadratically ordered subset. An explicit theoretical expression for the refolded lines is derived and the idea of a local world domain is explained.

In the next chapter, properties of biquadric fields are investigated in several ways. At first *flat* biquadric fields are searched. Only for $q = 3$ such a field is found. Furthermore the question arises whether representation matrices $(M, \overline{M})$ of a biquadric $\mathcal{B}$ are uniquely defined by the point set. For $q = 3$ and hyperbolic biquadrics in $\mathbb{P}^2\mathbb{F}_5$ indeed different representation matrices with the same point set exist. This ambiguity is removed when bigger prime numbers are considered, what will be investigated in terms of *Lorentz-Transformations*. Here it is shown that it is not possible to exchange single points or point sets between the two quadric partners of

[1] Where d is the projective dimension and q is a prime.
[2] Which are necessary to add non-trivial properties to the projective geometry.

a biquadric. The action on the point set can be interpreted as *rotational*, *axial-symmetric* or *point-symmetric* transformations.

Biquadric fields connect the points of the finite geometry to each other, encoded in neighborhood relations. These are studied by the dynamics of cellular automata, especially the *Game of Life*. The rules are adapted to finite geometries and biquadric neighborhoods, indeed changing the behavior of the dynamical system. For homogeneous biquadric fields three different phases in the density development are seen, while for inhomogeneous fields the behavior significantly depends on the chosen inhomogeneity. The three phases in the homogeneous case can be interpreted as a vanishing regime ($\rho_{\text{init}} \to 0$), a continuous regime (ρ_{init} does not change) and a plateau regime, where all initial densities ρ_{init} converge to the same stationary density ρ_{stat}. The theoretical treatment of the homogeneous case in terms of a cumulant expansion strengthens the simulation results. At a certain percentage of defects in the system the runtime until stationarity diverges, while the stationary density in the plateau phase jumps from $\rho_{\text{stat}} \approx 0.4$ to $\rho_{\text{stat}} \approx 0.27$.

Contents

1 Introduction - Unification of Space, Time and Matter?

> Two roads diverged in a yellow wood,
> And sorry I could not travel both
> And be one traveler, long I stood
> And looked down one as far as I could
> To where it bent in the undergrowth
>
> Robert Frost, *The Road Not Taken*

What is time? What is matter? Why do we live in 3 spatial dimensions? - Some of the most fundamental questions humans can ask remain unanswered by physics so far. But finite projective geometries constitute a promising, discrete world model for giving new answers to these and other questions.([Mec18], p.1)
Starting with Einstein's discovery of *General relativity* (GR), which describes the universe on large scales, and the development of quantum field theory (QFT), which shapes the world on microscopic scales, the search for a unification of both theories started. Both theories on their own reproduce so far all experimental results, but still have their own conceptual problems. Since the very beginning of science and nature observation, scientists were searching for some kind of a *Theory of Everything*, describing all phenomena we observe in a unified language - GR would then be the limiting case for big lengths scales (small quantum effects), while QFT should be reproduced for little length scale (quantum effects not negligible). Unfortunately, QFT and GR are based on fundamentally different mathematical structures and treat even the most fundamental quantities differently. In general relativity the geometry, described by the metric tensor, itself is a dynamical object influenced by the matter, which is present in the universe, described by the energy-momentum tensor. Therefore one

can say that this theory is local. In QFT the matter fields, from which the energy-momentum tensor has to be derived, are placed on a preexistent, continuous and differentiable 4-dimensional manifold equipped with a metric, called spacetime, resulting in a global theory (for a more detailed list of problems and unasked/unanswered questions, see [Mec18] starting at page 19).

In general two very basic ideas exist, how to construct a theory of quantum gravity - either by quantizing a continuous theory of spacetime (*top-down approach*) or by starting from a discrete theory of spacetime and develop the quantum features of particles from this intrinsic discreteness of the model (*bottom-up approach*).
Nowadays the two best-known approaches for quantum gravity are both *top-down*: String theory and Loop Quantum Gravity both try to quantize gravity starting from a continuous theory. Up to now both theories produce no experimentally verified results, therefore the search for a new theory goes on. Known *Bottom-up approaches* include the Twistor Theory[Pen06] or Causal Sets Theory [Bom+87], which also fail to make contact to the real world until now.
Another idea of a *bottom-up* theory is the background of this thesis: Equipping finite fields with spacetime properties, such that some intrinsic quantization is already given. But even more, this approach does not only quantize spacetime, but also tries to express matter as an emerging property of spacetime that is already inherent to the finite projective geometry, it tries a complete unification of space, time and matter, which historically already Descartes dreamed of ([Des70], p.85 and [Wey88], p.1) The idea of finite fields as a model for spacetime is not completely new to physics, but was historically often rejected in favor of continuous models, nevertheless many physicists as Y. Ahmavaara in the 1960s ([Ahm65]) or Paul Kustaanheimo and Gustaf Järnefelt ([JK49] and [Jär76]) saw and still see that finite fields could indeed be a suitable way to describe our world. Others are more critical concerning finite number systems, but still see the elegance finite projective geometries obey, as R. Penrose states in his book *The road to reality.*

> "Although such a standpoint [using finte fields] must be regarded as distinctly unconventional, it is not inherently incon-

> sistent. Indeed there has been a school of thought that the apparently basic physical role of the real-number system $\mathbb{R}$ is some kind of approximation to a 'true' physical number system which has only a finite number of elements." ([Pen06], p.357)

Clearly this thesis cannot accomplish the immense task of establishing a new theory based on finite fields, lots of work has already been done and maybe even more work will have to be done on this in the future. What this thesis does is to explore some properties of finite geometries as a framework for a discrete spacetime.

Encoding lengths and distances in the same fashion as in GR by using quadrics[1] leads to problems in finite fields, as one quadric does not have enough points on it to define a unit distance in each direction. This problem can be overcome with the introduction of a second quadric, which together with the first defines the so-called biquadric. Previous thesis on this topic found an explicit parametric form of such objects, the complete geometric structure of spacetime is defined if a biquadric is centered around every point - resulting in a biquadric field. This makes it possible to define local distances from every point in every possible direction by means of cross sections[2]. ([Las14] and [Mec17])
The chapters two and four of this thesis will mainly concentrate on the mathematical properties of finite projective geometries as a whole and especially of biquadric fields. Chapter 2 will introduce the concepts from a very general point of view, starting with the axiomatic definition of what a geometry is. By specifying ore and more components of the geometry we eventually arrive at finite projective geometries. From the definition of a single quadric the need for biquadrics will be made clear, before the whole finite projective geometry is equipped with biquadric fields.

The invariance symmetry group of biquadric fields was already analyzed in [Rei16], nevertheless the analysis will be expanded to the point set level in chapter 4 of this thesis. Furthermore we will analyze if biquadrics de-

[1] The quadric for the metric tensor $g_{\mu\nu}$ is the set $\mathcal{Q} = \{x \in \mathbb{R}^4 | g_{\mu\nu} x^\mu x^\nu = 0\}$.
[2] Which are the only projectively invariant property in projective geometries.

fine the neighborhood of a given spacetime point uniquely or if ambiguity exists. To get closer to a definition of curvature that is tremendously important in GR so-called *flat biquadric fields* are searched.

In chapter 5 the neighborhood relations of biquadric points are investigated in terms of cellular automata. This includes the questions: *How are the points connected? How do they influence each other when certain rules are imposed? How can the dynamics of such a system be described?* Cellular automata in general are a well-established tool to analyze dynamical systems that obey simple rules and can model complex physical behavior. The explicit relation of points in a finite geometry equipped with biquadric fields can be unintuitive, therefore a well-known cellular automata, the *Game of Life* was generalized to finite projective geometries. Our hope was to get some intuition for the behavior of biquadric fields, by comparing the results of the standard *Game of Life* with the projective one. Therefore homogeneous fields were studied, as well as inhomogeneous ones. This can also give some hint on the relation of homogeneous and non-homogeneous biquadric fields. For the theoretical explanation a cumulant expansion up to second order is done, to account for the strong correlations between the points due to the high connectivity of biquadric fields.

The project showed interesting new properties of biquadric fields as a metric background for spacetime. Nevertheless it also opened new questions concerning discrete spacetime. How this can be exactly used to model the physical world we observe every day has to be shown by further research.

2 Projective geometry and its properties

"To understand the universe, you must understand the language in which it is written, the language of Mathematics."

Galileo Galilei

The construction of a model for space-time with only finitely many elements and a projective geometry needs some non-trivial mathematical structures. The foundations of these shall be introduced during this chapter. We will start by introducing a general notion of geometry, concentrating especially on *projective geometries*. In the second part, finite fields, so-called *Galois-Fields* $\mathbb{F}_p$ are explained in more detail, before these two concepts are combined to *finite projective geometries* of arbitrary projective dimension d [1]. The case d=4 serves as our general space-time structure, but in this thesis the case d=2 is studied in more detail, as it is not as CPU-expensive in simulations and still gives good hints on the behavior in 4 dimensions.
Quite obviously one also needs some notion of neighborhood to define the relation between arbitrary points in the finite projective geometry, therefore *biquadrics* $(\mathcal{Q}_M, \mathcal{Q}_{\overline{M}})$ are introduced as a tool for measuring distances and lengths.

[1] It is important to mention already here, that projective dimension d means vector-space dimension $d+1$ - this will get clear in the following sections

2.1 Geometries and its axioms

2.1.1 General definitions of geometries

On the first pages of this section the axiomatic way towards geometries will be presented. To get intuition for this, it is important to think about what the heart of geometry is - it is the relation between points, lines, and in general hypersurfaces to each other. The abstract definition makes exactly this idea clear. ([BR04], p.1)

Definition 2.1.1. Geometry A geometry is a pair $\mathcal{G}$=(Ω, $\mathcal{I}$) of a set Ω and a relation $\mathcal{I}$ on Ω, which is *reflexive* and *symmetric*.

- $\forall\, x, y \in \Omega : (x, y) \in \mathcal{I} \rightarrow (y, x) \in \mathcal{I}$
- $\forall\, x \in \Omega : (x, x) \in \mathcal{I}$

The set Ω consists of all geometrically relevant objects, such as points, lines or hypersurfaces - depending on the dimension of the geometry. The relation $\mathcal{I}$ is called an *incidence relation* and clarifies the relation between two objects $x, y \in \Omega$. [2]
Analyzing geometries in more detail leads to the notion of flags, which are subsets of Ω whose elements are pairwise incident. ([BR04], p.3)

Definition 2.1.2. Flag $\mathfrak{F}$ Let $\mathcal{G} = (\Omega, \mathcal{I})$ be a geometry. A *flag* $\mathfrak{F}$ of $\mathcal{G}$ is a set of elements from Ω which are pairwise incident. A flag $\mathfrak{F}$ is called *maximal*, if there is no element $x \in \Omega \setminus \mathfrak{F}$, such that $\mathfrak{F} \cup \{x\}$ is also a flag.

This definition of a flag has a major advantage - it allows us to define a notion of geometries that allow for a segmentation of the large set Ω into several smaller disjoint sets Ω_i of certain objects. Therefore a third definition is needed. ([BR04], p.3)

Definition 2.1.3. Rank of a geometry A geometry $\mathcal{G} = (\Omega, \mathcal{I})$ is called to be of *rank* r if Ω can be segmented into disjoint sets Ω_i for $i \in \{1, ..., r\}$, such that every maximal flag of $\mathcal{G}$ contains exactly one element of each Ω_i. In this case the elements of Ω_i are called elements of *type* i.

[2] In most cases it is possible to translate $(x, y) \in \mathcal{I}$ as *"x is contained in (is incident with) y"*.

As an example it may be helpful to consider the known 3-dimensional space. Take a point x which is incident with a line l. Furthermore l shall be incident with a plane p, then $\{x, l, p\}$ is a maximal flag. As every other maximal flag also has 3 elements, the known 3-dimensional space is a geometry of rank 3. Following the definition we take as Ω_1 the set of all points, Ω_2 the set of all lines and as Ω_3 the set of all planes.
Being equipped with some geometrical structures it is now time to face a special type of geometries - namely *projective geometries*.

2.1.2 The axioms of projective geometry

The introduction of projective spaces may at first sight be counter-intuitive, as one of the used axioms (Veblen-Young axiom) excludes the existence of parallel lines, in the sense that every two lines which lie in the same 2-dimensional subspace intersect. Nevertheless, projective geometries are very general and fulfill a convenient duality between points and hyperplanes (lines), furthermore affine geometry (as we experience it in our daily life) is just a special case of the projective one. In the following we concentrate on geometries of rank 2 $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ where $\mathcal{P}$ is the set of all points and $\mathcal{L}$ is the set of all hyperplanes (lines). $\mathcal{I}$ is the *"lies into"*-incidence relation.

Definition 2.1.4. Projective plane ([BR04], p. 5–7) Let $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a geometry of rank 2. Then we call $\mathcal{G}$ a projective plane, if the following 4 axioms are fulfilled.

A1: For any pair of points p and q there exists always one line l that is incident with both of them: $\forall\, p, q \in \mathcal{P}\ \exists!\ l \in \mathcal{L} : (p, l) \in \mathcal{I} \wedge (q, l) \in \mathcal{I}$

A2: For any pair of lines l and g there exists always one point p that is incident with both of them: $\forall\, l, g \in \mathcal{L}\ \exists!\ p \in \mathcal{P} : (p, l) \in \mathcal{I} \wedge (p, g) \in \mathcal{I}$

A3: Every line is incident with at least three points.

A4: There are at least two different lines.

Axiom A2 excludes parallels from the projective geometry by definition. Nevertheless, we are used to the notion of parallelism from our everyday life, therefore the notion of an affine space, where parallels exist, shall also be introduced, but in a less formal way.

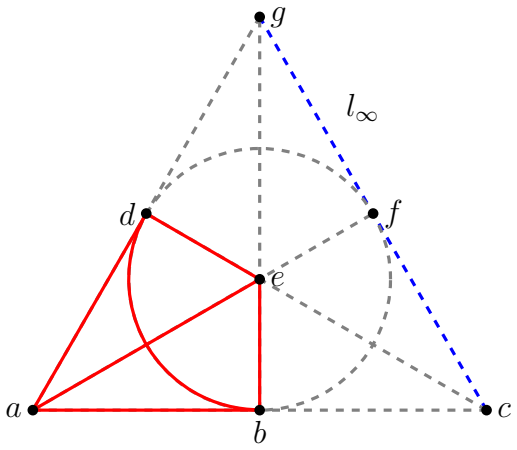


Figure 2.1: The smallest projective plane $\mathbb{P}^2\mathbb{F}_2$ for the prime number p=2. The red points and lines can be interpreted as an affine plane with 4 points and 4 lines. The red parallel lines $\overline{de}$ and $\overline{ab}$ intersect in the point c at infinity. Similarly $\overline{ad}$ and $\overline{be}$ meet in g and $\overline{ae}$ meets $\overline{bd}$ in f.

Remark 1. After this definition it is now straight forward to construct an affine plane from the projective plane. Therefore one has to exclude one line l_∞, the *line at infinity*, from $\mathcal{L}$ to get a new set of lines $\mathcal{L}' = \mathcal{L} \setminus \{l_\infty\}$, and also all points on that line from $\mathcal{P}$ to get $\mathcal{P}'$. This results in an affine plane - in contrast to projective planes now parallel lines indeed exist, because the line at infinity where they would intersect has been removed. In the inverse fashion it is possible to construct a projective plane from an affine plane by adding points where parallels intersect and connect them by a new line.

The smallest non-degenerate projective plane is called *Fano-plane*, which consists of seven points and seven lines. One can interpret it as an affine square, to which an additional *line at infinity* has been added, that contains all intersections of parallel lines. The notion of projective planes is very useful to define projective spaces of higher dimensions, where higher dimensional subsets (not only points and lines) can exist.

Definition 2.1.5. Projective space ([Bat97], p. 60 We call $\mathcal{G}$ a projective space, if and only if every two-dimensional subspace fulfills the axioms of

a projective plane.

Remark 2. The notion of projective spaces can also be introduced in another axiomatic way, using the **Veblen-Young** axiom. This axiom replaces A2 in the definition of a projective plane. ([BR04], p. 5–7)

A1: If p and q are two different points, there is exactly one line l, that is incident with p and q. This line is called $\overline{pq}$.

A2: **Veblen-Young** If a, b, c, d are four points, such that $\overline{ab}$ intersects $\overline{cd}$, then $\overline{ac}$ also intersects $\overline{bd}$.

A3: Every line is incident with at least three points.

A4: There are at least two different lines.

Remark 3. By using only the first axiom A1 it is already possible to proof that two different lines cannot have more than one intersection in projective spaces. Furthermore [BR04] states that the **Veblen-Young** axiom is a clever way to to state that two lines in a plane intersect - without knowing what a plane in detail is. But in contrast to projective planes there can also be lines that do not intersect, as long as they are not incident with the same plane. Another consequence of this axiom is the following: If two lines $\overline{ab}$ and $\overline{cd}$ do not intersect, then also $\overline{ac}$ and $\overline{bd}$ do not intersect.

Taking a close look at the axioms of *projective planes*, it gets obvious that lines and points are treated symmetric, as axiom A1 and A2 are dual to each other. That means both axioms state the same thing, just the notion of points and lines is exchanged.
By exchanging the words *points* and *lines* in a statement S about rank 2 geometries, one gets the so-called *dual statement* $S^{\triangle}$.
In the following theorem we will see, that projective planes indeed satisfy the duality principle. ([BR04], p. 9 and [RO09], p. 30)

Theorem 2.1.1. ***Duality for projective planes*** *If S is a statement which is true for all projective planes, then $S^{\triangle}$ is also true for all projective planes.*

Remark 4. The duality of points and lines as it is shown here is only true for projective planes. If the rank increases[3] an analogous statement can be proven for the points of the geometry and the (d-1)-dimensional hyperplanes. Therefore these two can be exchanged in every statement and it remains true. (vgl. [Las14], p. 17)

2.2 Construction of finite fields

The axiomatically constructed geometries so far lack an important property - up to now they only consist of abstract points $\mathcal{P}$, lines $\mathcal{L}$ and an incidence relation $\mathcal{I}$ between objects from the two sets. By now we have no possibility to do calculations with them and use numbers, as we are used to it from coordinatized continuous geometries. Therefore now the notion of *Galois-Fields* shall be introduced, these are the finite sets on which we will base our coordinatization of *finite projective geometries*. Furthermore the whole procedure of coordinatization using equivalence relations shall be introduced.

2.2.1 From groups to Galois-fields and vector spaces

Some basic algebraic notion have to be introduced in order to understand the construction of finite fields. The first notion is that of a group.

Definition 2.2.1. Group([Ros09], p. 11–12) A *group* is a non-empty set $\mathfrak{G}$ together with a binary operation $\circ : \mathfrak{G} \times \mathfrak{G} \mapsto \mathfrak{G}$ which satisfies the following axioms:

- *Well-definedness*: $\forall x, y \in \mathfrak{G} : x \circ y \in \mathfrak{G}$
- *Associativity*: $\forall x, y, z \in \mathfrak{G} : (x \circ y) \circ z = x \circ (y \circ z)$
- *Neutral element*: $\forall x \in \mathfrak{G} \exists! e \in \mathfrak{G} : x \circ e = e \circ x = x$
- *Inverse element*: $\forall x \in \mathfrak{G} \exists! x' \in \mathfrak{G} : x \circ x' = x' \circ x = e$.

[3] Which can happen if we use higher dimensional projective spaces.

If $\circ$ is also commutative[4] then the group $(\mathfrak{G}, \circ)$ is called *abelian group*.

Groups have only one inner operation between the group elements, but it is possible to introduce a second operation and therefore get additional structure. If both operations have certain properties, one calls this structure a field.

Definition 2.2.2. Field([KB18], p. 352) A non-empty set $\mathbb{F}$ together with two binary operations $+ : \mathbb{F} \times \mathbb{F} \mapsto \mathbb{F}$ and $\cdot : \mathbb{F} \times \mathbb{F} \mapsto \mathbb{F}$ is called field, if

- $(\mathbb{F}, +)$ is an abelian group
- $(\mathbb{K} \setminus \{0\})$ is an abelian group
- $\forall x, y, z \in \mathbb{F} : x \cdot (y + z) = (x \cdot y) + (x \cdot z) \wedge (x + y) \cdot z = (a \cdot z) + (y \cdot z)$

The field is called finite if the underlying set $\mathbb{F}$ has only finitely many elements. Finite fields can be constructed by taking the modulo-operation on the integers.

Remark 5. **Construction of finite fields**([Rei16], p. 8) Let $q \in \mathbb{Z}$ be prime. We define on $\mathbb{Z}$ the equivalence relation

$$\forall n, m \in \mathbb{Z} : n \sim m \leftrightarrow \exists k \in \mathbb{Z} : n - m = k \cdot q.$$

All elements that differ only by multiples of a the prime number q are identified. Then the set of all equivalence classes $\mathbb{Z}/q\mathbb{Z} := \{[n]_\sim | n \in \mathbb{Z}\}$ together with the addition[5]

$$+_{\text{mod q}} : \mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \mapsto \mathbb{Z}/q\mathbb{Z}$$

and the multiplication[6]

$$\cdot_{\text{mod q}} : \mathbb{Z}/q\mathbb{Z} \setminus \{0\} \times \mathbb{Z}/q\mathbb{Z} \setminus \{0\} \mapsto \mathbb{Z}/q\mathbb{Z} \setminus \{0\}.$$

[4] *Commutativity*: $\forall x, y \in \mathfrak{G} : x \circ y = y \circ x$.
[5] $[n]_\sim +_{\text{mod q}} [m]_\sim := [n + m]_\sim$.
[6] $[n]_\sim \cdot_{\text{mod q}} [m]_\sim := [n \cdot m]_\sim$.

Then $(\mathbb{Z}/q\mathbb{Z}, +_{\text{mod q}}, \cdot_{\text{mod q}})$ is a field with q elements and called $\mathbb{F}_q$ in the following. If q was not prime, $\mathbb{F}_q$ would not be free of zero divisors and therefore would not be a field.([Ros09], p. 40)
Furthermore an important theorem states that up to isomorphisms only one finite field of order q exists, these are the $\mathbb{F}_q$ as we constructed them here.([Ros09], p. 127) It holds: $F_q = \{0, 1, ..., q-1\} = \{-\frac{q-1}{2}, ..., 0, ..., \frac{q-1}{2}\}$. During the whole thesis we restrict to fields of order q where q is prime. It would be possible to construct finite fields for every $p = q^n$ where q is prime and $n \in \mathbb{N}$. This would lead to far more complex calculations and is therefore not taken into account at this stage of the theory.

As a last step before the construction of finite projective geometries, we have to define the notion of a vector space. These spaces are used as the space where all points live in, therefore their properties are of tremendous importance for the whole theory.

Definition 2.2.3. Vector space([Str12], p. 125) Let $\mathbb{F}$ be a field and V a set. For every two elements $a, b \in V$ there be a unique element $(a \oplus b) \in V$[7] and for every $\lambda \in \mathbb{F}$, $a \in V$ there be a unique element $\lambda \bullet a \in V$[8]. Then the triple $(V, \oplus, \bullet)$ is called a *vector space* over $\mathbb{F}$ if $(V, \oplus)$ is an abelian group, and if $\bullet$ fulfills

- For every $a \in V$ there is a neutral element $\lambda_0 \in \mathbb{F}$, such that $\lambda_0 \bullet a = a \bullet \lambda_0 = a$
- For every two elements $\lambda, \mu \in \mathbb{F}$ and every $a \in V$ it holds $\lambda \bullet (\mu \bullet a) = (\lambda \cdot \mu) \bullet a$
- For every element $\lambda \in \mathbb{F}$ and every two elements $a, b \in V$ it holds $\lambda \bullet (a \oplus b) = (\lambda \bullet a) \oplus (\lambda \bullet b)$
- For every two elements $\lambda, \mu \in \mathbb{F}$ and every element $a \in V$ it holds $(\lambda + \mu) \bullet a = (\lambda \bullet a) \oplus (\mu \bullet b)$

All of these properties are already known from the frequently used vector space $\mathbb{R}^3$, but generalized here to all possible underlying fields.

[7] $\oplus : V \times V \mapsto V$ is an inner binary operation called *vector addition*.
[8] $\bullet : V \times \mathbb{F} \mapsto V$ is an outer binary operation called *scalar multiplication*.

Remark 6. For a given field $\mathbb{F}$ the n-fold Cartesian product $\mathbb{F}^d := \mathbb{F} \times ... \times \mathbb{F}$ together with the addition and scalar multiplication

$$(x_1, ...x_d) + (y_1, ..., y_d) = (x_1 + y_1, ..., x_d + y_d)$$
$$\lambda \cdot (y_1, ..., y_d) = (\lambda \cdot y_1, ..., \lambda \cdot y_d)$$

defines a vector space over $\mathbb{F}$ as one can easily check by checking the properties of addition and multiplication.

2.2.2 Homogeneity equivalence relation and coordinatization

After the introduction of vector spaces as well as finite fields we are only one step away from finite projective geometries. At first we introduce general projective geometries and later just concentrate on the finite case. The procedure that is presented now has the big advantage that the projective space is already given in terms of coordinates what makes calculations possible.

Definition 2.2.4. Homogeneity relation([Ric11], p. 47) Let $\mathbb{F}$ be any field and $\mathbb{F}^d$ a vector space above it. Then the equivalence relation

$$\forall\, p, q \in \mathbb{F}^d \setminus \{0\} : p \sim q \leftrightarrow \exists\, \lambda \in \mathbb{F} \setminus \{0\} : p = \lambda \cdot q$$

is called homogeneity.

This equivalence relation identifies all elements in the given vector space that are multiples of each other. Therefore in terms of the well-known euclidean vector space all points that lie on the same line through the origin would be identified. From a $d+1$ dimensional vector space, one can now construct a n-dimensional projective space as a quotient set.

Definition 2.2.5. Projective space([Las14], p. 27) Let $\mathbb{F}^{d+1}$ be a vector space and $\sim$ the homogeneity equivalence relation, then the d-dimensional projective space over $\mathbb{F}$ is defined by

$$\mathbb{P}^d\mathbb{F} := (\mathbb{F}^{d+1} \setminus \{0\})/\sim = \{[p]_\sim | p \in \mathbb{F}^{d+1} \setminus \{0\}\}.$$

Considering what the homogeneity equivalence relation does, it gets reasonable that the dimension of the projective space is smaller than the dimension of the vector space from which it is constructed. As already said, lines are identified with one single point, therefore a dimensionality reduction occurs. If the vectors are written in components, they still have $(d+1)$ components, but one of these components can always be normalized to zero or one. In this thesis we always norm the last component of the vectors. These equivalence classes of points are called their *homogeneous coordinates* and simplify many calculations. The projective lines, as the dual space $\mathbb{P}^d\mathbb{F}^*$ are constructed analogously and therefore can be normalized in the same way as the points.
One remaining question is how one can determine incidence of points and lines or calculate the line that connects two points within this coordinated framework to check whether the axioms of projective geometries are fulfilled. This works by the means of dot products[9] and generalized cross products.

Definition 2.2.6. Cross product([Las14], p. 30) The cross product is defined by

$$X : \mathbb{P}^d\mathbb{F} \times ... \times \mathbb{P}^d\mathbb{F} \mapsto \mathbb{P}^d\mathbb{F}^*$$
$$X : \mathbb{P}^d\mathbb{F}^* \times ... \times \mathbb{P}^d\mathbb{F}^* \mapsto \mathbb{P}^d\mathbb{F}$$

and therefore transforms points into their dual objects and vice versa. It is given by

$$X(v_1, ...v_n) := \det \begin{pmatrix} — & v_1 & — \\ & \vdots & \\ — & v_n & — \\ \hat{e}_1 & ... & \hat{e}_{n+1} \end{pmatrix}$$

In 2 projective dimensions this definitions boils down to the known cross product, but in the given form it can also be used in higher dimensions.

[9] $\circ : \mathbb{P}^d\mathbb{F} \times \mathbb{P}^d\mathbb{F}^* \mapsto \mathbb{F}$ defined by $p \circ l := p^i l_i$.

Now it can be shown that the set-theoretical containedness incidence relation is equivalent to the case that the dot product between hyperplane and point is vanishing. Therefore for a given line l, the points p_j on it are given as the solutions of the linear equation $l \circ p_j$. It can be shown that for finite fields of order q this equation has $(q+1)$ solutions. The lines that go through a given point can be calculated in the same manner.[10]
To connect two points by a line the cross product of the two is calculated. In higher dimensions one calculates the cross product of more than two points and gets higher-dimensional hypersurfaces as the analogue to lines in 2 dimensions. The point in which hypersurfaces intersect can also be calculated by taking their cross product. ([Las14], p. 31).
By now a lot of things about projective geometries have been said, and we are now able to define a finite projective geometry $\mathbb{P}^n\mathbb{F}_p$ as a projective geometry as we discussed it above, but with a finite field $\mathbb{F}_q$ as the underlying algebraic object. This has the consequence that many quantities are countable now. For example in 2 dimensions there are q^2+q+1 many points in the projective plane, q^2 of these are affine points and $q+1$ are on the line at infinity. Furthermore every line consists of $q+1$ points and every point is contained in $q+1$ lines. Here one can again see the duality between points and lines in projective planes.

2.3 Equipping finite projective geometry with biquadric fields

By now the whole framework of finite projective geometries was introduced. But without further structure this framework is not suitable for a physical world model, as at least a notion of neighborhood, distance or length should be defined. Therefore additional 'metric' structure is necessary. To examine this problem was the task of Alexander Laska's Master's thesis, from which a big part of the following results is taken.
By using finite projective spaces the notion of neighborhood, as it is known from the real numbers, is lost. In continuous projective geometries this

[10] Therefore also $q+1$ lines go through each point.

problem is solved by introducing a quadric $\mathcal{Q}$ around every point and defining distance and neighborhood in terms of the cross-ratio s^2, which is the only projectively invariant distance measure. In finite projective geometries one quadric is not enough, as it does not contain points in each direction from the center point. Therefore a second quadric is introduced, such that on every line through the center two points intersect the biquadric, defining the neighbors of p_c. These are defined to have unit distance and one uses again the cross-ratio to calculate projectively invariant distances. But still this does not resemble our everyday idea of distance, because the periodicity of the finite field comes into play, as can be seen in fig. 3.1. ([Rei16], p. 12 and [Mec17], p. 3)
Before biquadrics are introduced in more detail, one should focus on the easier notion of single quadrics, which are defined as the zero set of a quadratic equation. For every quadric an assignment of the form

$$\mathfrak{Q}_M(p) = p^t M p = M_{ij} p^i p^j$$

exists and is called a quadratic form[11] Here M is the projective matrix describing the specific quadratic form $\mathfrak{Q}$ corresponding to the quadric $\mathcal{Q}$, often called the *representation matrix*. ([RO09], p. 58 and [Las14], p. 37)

Definition 2.3.1. Projective matrix Let $\mathbb{F}^{d\times d}$ be the set of all $d \times d$-matrices over the field $\mathbb{F}$. The projective matrices are given by the same equivalence relation as before, as a quotient space

$$\mathbb{P}^{d\times d}\mathbb{F} = \mathbb{F}^{(d+1)\times(d+1)} / \sim$$

Therefore all projective matrices are unique up to multiples, which is an equivalent definition to that of projective points. One component can always be normalized to 1.

[11] A quadratic form $\mathfrak{Q}$ is a map $\mathfrak{Q} : V \to \mathbb{F}$ from a vector space V into the underlying field $\mathbb{F}$. It has the following properties

- $\forall v \in V : \forall k \in \mathbb{F} : \mathfrak{Q}(k \cdot v) = k^2 \mathfrak{Q}(v)$
- The map $\mathfrak{B} : V \times V \to \mathbb{K}$ defined by $\mathfrak{B}(v, w) := \mathfrak{Q}(v + w) - \mathfrak{Q}(v) - \mathfrak{Q}(w)$ is symmetric and bilinear.

Definition 2.3.2. Quadrics Let $\mathfrak{Q}_M$ be a quadratic form. The assigned projective quadric is defined by

$$\mathcal{Q}_M = \{p \in \mathcal{P} : p^t M p = 0\}.$$

The representation matrix of a quadratic form can, without loss of generality, be taken to be symmetric[12]. This can be seen straight forward if it is split into a symmetric part M_s and an antisymmetric part M_a. ([Las14], p. 39)

$$M = \frac{1}{2}(M + M^t) + \frac{1}{2}(M - M^t) = M_s + M_a \tag{2.1}$$

$$M_{ij}p^i p^j = M_{s,ij}p^i p^j + M_{a,ij}p^i p^j \tag{2.2}$$

$$= M_{s,ij}p^i p^j. \tag{2.3}$$

By inserting the explicit expression for the antisymmetric part $M_{a,ij}$ of M one sees fast, that it indeed vanishes if it is contracted with $p^i p^j$. The representation matrix can be interpreted as a $\binom{0}{2}$-tensor[13], which can take two $\binom{1}{0}$-tensors[14] and produce a number. By contracting the representation matrix with just one point, the result is a $\binom{0}{1}$-tensor $l_j = M_{ij}p^i$ which can be identified with a projective hyperplane.[15] This special hyperplane[16] is called the *polar hyperplane* of the point p and the matrix M.

Definition 2.3.3. Polar hyperplane ([Las14], p. 38) For a quadric $\mathcal{Q}_M$ the *polar hyperplane* is defined as the image of the point $p \in \mathcal{P}$ under the matrix $M : \mathcal{P} \to \mathcal{H}$, where $\mathcal{H}$ is the set of all $d-1$-dimensional hyperplanes:

$$\mathrm{pol}_M(p) := Mp$$

[12] A matrix is called symmetric if $M^t = M$.

[13] A $\binom{p}{q}$-tensor maps p elements of the dual vector space V^* and q elements of the vector space V linearly into the underlying field $\mathbb{K}$.

[14] These $\binom{1}{0}$-tensors are just the points of the projective space.

[15] This identification makes sense, as $p^t l = p^i l_i \in \mathbb{F}_q$ is a number and can therefore be seen as a $\binom{0}{0}$-tensor. Then the scalar product (incidence relation) between point and hyperplane is nothing else but a contraction of a $\binom{1}{0}$-tensor and a $\binom{0}{1}$-tensor.

[16] In 2-dimensional projective planes it is just a line.

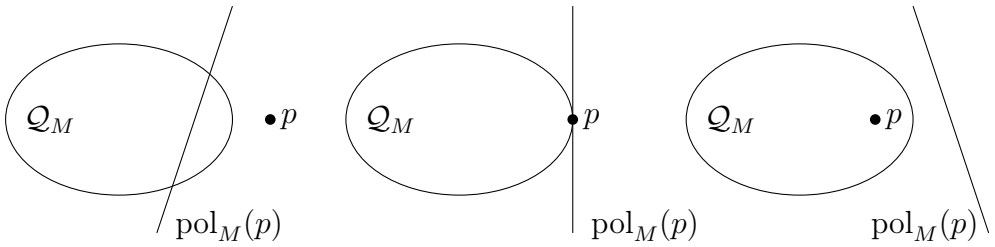


Figure 2.2: The relative position of a point and the quadric is determined by the intersections of its polar with the quadric. In the first picture the point is outside the quadric, as $|\mathrm{pol}_M(p) \cap \mathcal{Q}| > 1$. In the second sketch $|\mathrm{pol}_M(p) \cap \mathcal{Q}| = 1$, therefore the polar is a tangent and the point is on the quadric. The third figure shows the behavior if $|\mathrm{pol}_M(p) \cap \mathcal{Q}| = 0$. The point is then inside the quadric.

Remark 7. The concept of polar hyperplanes makes it possible to decide whether a point lies inside, on or outside a quadric. A point is said to be inside of $\mathcal{Q}$ if $|\mathrm{pol}_M(p) \cap \mathcal{Q}| = 0$, on the quadric if $|\mathrm{pol}_M(p) \cap \mathcal{Q}| = 1$ and outside if $|\mathrm{pol}_M(p) \cap \mathcal{Q}| > 1$ ([Ale12], p. 21-23).

So far we have dealt with single quadrics in finite projective spaces, and have said nothing about the real aim of this chapter: To define a unit distance in every possible direction from every possible point. Every point in a finite projective plane is incident with $q + 1$ lines, defining $2(q + 1)$ different directions (positive and negative). Identifying the intersections of the lines through the center point and the quadric as unit distances is not possible, as there are not enough points on a single quadric. A single quadric consists of $q+1$ points, which are point-symmetric to the center.[17] Therefore only half of the liens through the center intersect the quadric. The way out of this problem is the introduction of a second complementary quadric centered at the same point. This quadric should have its points exactly on those lines that do not intersect the first quadric, such that every line now intersects the biquadric twice. ([Mec17], p.2)

[17] That can be easily shown, because if $p^t M p = 0$, then also $(-p^t)M(-p) = p^t M p = 0$.

Definition 2.3.4. Biquadrics ([Las14], p. 54) For a point $p_c \in \mathbb{P}^d\mathbb{F}_q$, a biquadric $\mathcal{B}_{p_c}$ is a pair of two quadrics $(\mathcal{Q}_{\mathcal{M}}, \mathcal{Q}_{\overline{M}})$, such that every line through p_c intersects the points of $\mathcal{B}_{p_c}$ twice. The point p_c is called *center point*.

In analogy to the definition of polar hypersurfaces, one defines a special hypersurface with respect to each quadric, namely the distinguished *hypersurface at infinity* h_∞. This is done by multiplying all possible centerpoints[18] into the two representation matrices M and $\overline{M}$. Even if it is not clear from the definition we will in the following restrict to biquadrics that fulfill

$$\mathrm{pol}_M(p_c) = \mathrm{pol}_{\overline{M}}(p_c). \text{ ([Rei16], p. 16)}$$

In fact it is not true that every biquadric determines a unique center point p_c, there are already cases known where a quadric pair has one, two or four center points and still defines a biquadric, such that every center point fulfills the biquadric properties. This results also in a bigger number of hyperplanes at infinity. In the following we restrict also here to the case that only one center point exists, because for this [Las14] has shown that a parametrized construction algorithm exists. Nevertheless, during this thesis we will also search for non-unique biquadrics and find a new species with 3 center points which exists only for $q = 5$.
Following an idea of Klaus Mecke the parametrized form for the standard center point and the standard hyperplane at infinity is given by:

Definition 2.3.5. Parametrized form of biquadrics ([Las14], p. 56) Let $(\mathcal{Q}_M, \mathcal{Q}_{\overline{M}})$ be a biquadric for the center point $p_c = (0, ..., 0, 1)^t$ and the hypersurface at infinity $h_\infty = (0, ..., 0, 1)^t$. Then a $q \in \overline{\mathfrak{Q}}$[19] and a matrix $A \in \mathbb{F}^{d\times d}$ exists, such that the pair of matrices

$$(M, \overline{M}) = \left(\begin{pmatrix} A & \vec{0} \\ \vec{0}^t & 1 \end{pmatrix}, \begin{pmatrix} qA & \vec{0} \\ \vec{0}^t & 1 \end{pmatrix} \right)$$

[18] Up to now it is not clear how many center points exist for a given quadric pair and the definition says nothing about that.

[19] An element $a \in \mathbb{F}_q$ is called a non-square if no element $b \in \mathbb{F}_q$ exists, such that $b^2 = a$. There are $\frac{q-1}{2}$-many such numbers for q prime. The set is denoted by $\overline{\mathfrak{Q}}(\mathbb{F}_q)$.

is a pair of representation matrices of the biquadric $\mathcal{B}_{(M,\overline{M})} = (\mathcal{Q}_M, \mathcal{Q}_{\overline{M}})$.

Remark 8. This definition only applies to the standard center point and the standard hypersurface, but can be easily generalized to all combinations of hypersurfaces and center points. The factor q is used to distinguish between the two used quadrics and has to be a non-square.

So far it is known that two different types of biquadrics exist in projective geometry called *elliptic* and *hyperbolic*. They are distinguished by the relative position of the center point regarding the single quadrics.

- If the centerpoint p_c lies outside both quadrics the resulting quadric is biquadric is called *hyperbolic*.
- If the centerpoint p_c lies inside both quadrics the resulting biquadric is called *elliptic*.

In 2 projective dimensions a hyperbolic biquadric has 2 points on the line at infinity[20] and $2(q-1)$ points in the affine plane. Elliptic biquadrics do not intersect the line at infinity at all, therefore have $2(q+1)$ points in the affine plane. In the standard case of $p_c^t = (0,0,1)$ and $l_\infty = (0,0,1)$ one can determine easily if a biquadric is hyperbolic or elliptic by looking at the submatrix A from above. If $-\det(A) \in \mathfrak{Q}(\mathbb{F}_q)$ the biquadric is hyperbolic, otherwise elliptic. ([Ale12], p. 29)
To define a biquadric for arbitrary center points p_c and arbitrary hyperplanes at infinity one can either use appropriate automorphism as introduced in the next chapter of the following mechanism at least in 2 projective dimensions [21]

Definition 2.3.6. Biquadric for arbitrary p_c and l_∞, ([Las19]) A biquadric $\mathcal{B}_{(p_c,l_\infty)}$ can be calculated by choosing an arbitrary hyperplane l_∞, two arbitrary lines l_1 and l_2 that are incident with p_c (that means $l_1 \times l_2 = p_c$), a symmetric, invertible submatrix A, a square $q \in \mathfrak{Q}(\mathbb{F}_q)$ and

[20] Each of the two quadric partners has two points at infinity, but they are the same for both quadrics and therefore can be identified in biquadrics. The single quadrics still have $q+1$ points in total.
[21] The generalization to higher dimensions should work similarly.

a non-square $\bar{q} \in \overline{\mathfrak{Q}}(\mathbb{F}_q)$. Then the two representation matrices are given by

$$M = qHAH^t + l_\infty^t l_\infty \tag{2.4}$$
$$\overline{M} = \bar{q}HAH^t + l_\infty^t l_\infty \tag{2.5}$$

where

$$H = \begin{pmatrix} | & | \\ l_1 & l_2 \\ | & | \end{pmatrix}.$$

There is not only a single biquadric for each pair of center point and hyperplane, but some arbitrarily chosen parameter combinations in the formula above give the same biquadric what makes their enumeration complicated. More on this topic is said in the fourth chapter when we search for flat biquadric fields.
With the definition of a single biquadric at a single point it is now possible to define the neighborhood of this point and therefore all points that can be reached within one step (distance 1).

Definition 2.3.7. Neighborhood of p_c All points in $\mathcal{B}_{p_c}$ are called *neighbors* of p_c and are said to have distance 1. The set

$$\mathcal{B}_{p_c} = \{p \in \mathbb{P}^d\mathbb{F}_q : p^t M p = 0 \vee p^t \overline{M} p = 0\}$$

is called the *set of all neighbors*.

For reasonable physics it should be possible to define such a neighborhood for every point in the projective geometry. Such a structure is called a biquadric field ([Las14], p. 68).

Definition 2.3.8. Biquadric field For a projective geometry $(\mathcal{P}, \mathcal{H}, \mathcal{I})$ a *biquadric field* is a bijective map $\mathcal{B} : \mathcal{P} \mapsto \mathfrak{B}$, such that for all $p \in \mathcal{P}$, $\mathcal{B}(p)$ is a biquadric with center point p. Here $\mathfrak{B}$ denotes the set of all biquadrics.

Remark 9. The notation $\mathcal{B}$ without indices denotes a biquadric field, whereas $\mathcal{B}_{p_c}$ denotes the biquadric centered at p_c. If a biquadric has several center points, they will be separated like $\mathcal{B}_{p_{c1},\ldots,p_{cn}}$.

From the definition of biquadric fields it is clear that the biquadrics mapped to different points encode different neighborhoods and therefore different unit lengths at each point - this could be used to define a notion of curvature in the next steps of the theory. After the introduction of biquadric fields, we are now able to proceed to transformations in projective geometries.

2.4 Transformation of projective objects and automorphisms

By now we have a solid knowledge about all important objects in projective geometries, but what we are still missing is how to transform them. Therefore it is important to talk about automorphisms of projective spaces, called *projectivities*. An automorphism of an algebraic structure $\mathcal{A}$ (i.e. group $(\mathfrak{G}, \circ)$, ring $(\mathfrak{R}, \circ, +)$ or vector space $(V, \bullet, \oplus)$) is a map $\Phi : \mathcal{A} \mapsto \mathcal{A}$ that is bijective[22] and a homomorphism[23].

Definition 2.4.1. Projectivity An automorphism $\pi : \mathbb{P}^d\mathbb{F} \mapsto \mathbb{P}^d\mathbb{F}$ is called *projectivity*. The set of all projectivities is called $\mathrm{Aut}(\mathbb{P}^d\mathbb{F})$.

What makes projectivities interesting for our investigation of projective spaces is the fact, that they preserve incidence due to their property of being a homomorphism. This means it makes no difference whether one first calculates inside a projective space and then applies the projectivity or first applies the projectivity and then calculates with the transformed objects. The result will be the same.

[22] A bijective map $\Phi : \mathcal{A} \mapsto \mathcal{B}$ is a map that is injective ($\forall x, y \in \mathcal{A} : \Phi(x) = \Phi(y) \Rightarrow x = y$) and surjective ($\forall\, b \in \mathcal{B} \exists\, a \in \mathcal{A} : \Phi(a) = b$). If a map is bijective an inverse map Φ^{-1} exists.

[23] This definition is valid for groups, but can easily be generalized to other structures. If $\mathfrak{G}$ and $\mathfrak{H}$ are groups, and $\phi : \mathfrak{G} \mapsto \mathfrak{H}$ is a map between them, then ϕ is called a homomorphism, if $\forall\, g_1, g_2 \in \mathfrak{G} : \phi(g_1 \circ_{\mathfrak{G}} g_2) = \phi(g_1) \circ_{\mathfrak{H}} \phi(g_2)$. ([KB18], p. 349) If an algebraic structure has more binary operations, this condition has to hold for every operation.

Remark 10. Every projectivity $\pi \in \mathrm{Aut}(\mathbb{P}^d\mathbb{F}_q)$ can be written as an invertible projective matrix $\Pi_\pi \in \mathbb{P}^{d\times d}\mathbb{F}_q$ and it holds $\pi(p) = \Pi_\pi \circ p$. This theorem is called *fundamental theorem of projective geometry.* (proof and theorem for $d = 2$ in [Ric11], p. 62).

In the fashion of our tensor calculus for biquadrics, polars and points on them, projectivities can be identified with $\binom{1}{1}$-tensors, as they map points on points.

Definition 2.4.2. Transformation behavior of points([Las14], p.41) Let $\Pi : \mathbb{P}^d\mathbb{F} \mapsto \mathbb{P}^d\mathbb{F}$ be a projectivity. Then points $p \in \mathbb{P}^d\mathbb{F}$ transform according to

$$p' = \Pi p = \Pi^i_j p^j \hat{e}_i \tag{2.6}$$

where $\hat{e}_i$ is a basis of the vector space underlying the projective space.

From the fact that automorphisms preserve incidence one can immediately conclude how hyperplanes, quadrics and biquadrics transform under projectivities.

$$\begin{aligned} p^t l &\overset{!}{=} (p')^t l' \\ &= (\Pi \cdot p)^t l' \\ &= p^t \Pi^t l' \\ &\Leftrightarrow l = \Pi^t l' \Rightarrow l' = \Pi^{-t} l \end{aligned}$$

Hyperplanes have to transform with the inverse transposed matrix. In terms of quadrics preserving incidence can be translated to: If a point lies on the original quadric, the transformed point has to be incident with the transformed quadric. This is calculated as follows:

$$\begin{aligned} p^t M p &\overset{!}{=} p'^t M' p' \\ &= (\Pi \cdot p)^t M' (\Pi \cdot p) \\ &= p^t (\Pi^t M' \Pi) p \\ &\Leftrightarrow M = \Pi^t M' \Pi \Rightarrow M' = \Pi^{-t} M \Pi^{-1}. \end{aligned}$$

And the inverse always exists, because projectivities are represented by regular and therefore invertible matrices.
Analogous to projectivities, one can also search for automorphisms of the affine space, these are called *affinities.*

Definition 2.4.3. Affinities ([Las14], p 43) Affinities $\alpha : \mathbb{F}^d \mapsto \mathbb{F}^d$ are automorphisms of the affine space, mapping linear subspaces $\mathcal{U}$ onto linear subspaces $\alpha(\mathcal{U})$. For all affinities there is a matrix $A \in \mathbb{F}^{d\times d}$ and a translation vector $\vec{t} \in \mathbb{F}^d$, such that

$$\alpha_{A,\vec{t}}(\vec{p}) := A\vec{p} + \vec{t}.$$

Examples of such affinities are translations or rotations. To give the explicit form in terms of homogeneous coordinates of affinities in the affine plane of a given projective plane it is important to do this with respect to the chosen line at infinity. This procedure is called *homogenization* and *dehomogenization* and described in detail in [Las14] starting at p. 43. For the standard case $h_\infty = (0, 0, 1)^t$ the form of a translation is given by

$$T_{\vec{t}} = \begin{pmatrix} \mathbb{1}_2 & \vec{t} \\ \vec{0}^t & 1 \end{pmatrix}$$

This will later be used to construct flat biquadric fields in the affine space. The last 20 pages were filled with important knowledge about finite projective geometries, all of this will be used in the next chapters of this thesis, some things might be directly visible, others are just in the background, but nevertheless necessary for the whole framework to work.

3 Refolding of projective lines into local world domains

Before biquadrics and biquadric fields are studied, we will investigate some properties of the local world domain, which is of tremendous importance in the construction of a physical world model starting from finite projective geometries, as this chapter will show.

3.1 Refolding projective lines into local world domains

Calculating distances by using the cross-ration as defined in [Mec17] in terms of biquadrics includes calculating squares of numbers. But when squares have to be calculated in finite geometries, it is very likely that the periodicity of the finite field destroys the ordering of the numbers. Nevertheless these second order objects (biquadrics) are needed to add more complex structure to $\mathbb{P}^d\mathbb{F}_q$. ([Mec17], p. 5) Consider the following example.
Let $a < b \in \mathbb{F}_q$, $a < \sqrt{q}$ and $b > \sqrt{q}$. Then

$$a^2 < q$$
$$b^2 > q.$$

Now b^2 feels the periodicity of the finite field and it could easily be that $b^2 \mod q < a^2 \mod q = a^2$, thus violating a reasonable quadratic order in the finite space.
Therefore one restricts the physical domain of points, such that some kind of order (quadratic, linear, ...) can be introduced inside this domain. One idea is to use a quadratically ordered local domain $\mathbb{L}$ in the sense, that $\forall\ x \in \mathbb{L} : x^2 < q$, so that these points do not feel the periodicity of the

prime field when they are squared. Therefore $\mathbb{L} := [-\lfloor \sqrt[\mathbb{R}]{q} \rfloor, \lfloor \sqrt[\mathbb{R}]{q} \rfloor]$ is used. Fig. 3.1 shows that the quadratically ordered local world domain is only a small part of the whole projective space. One idea to use the properties of

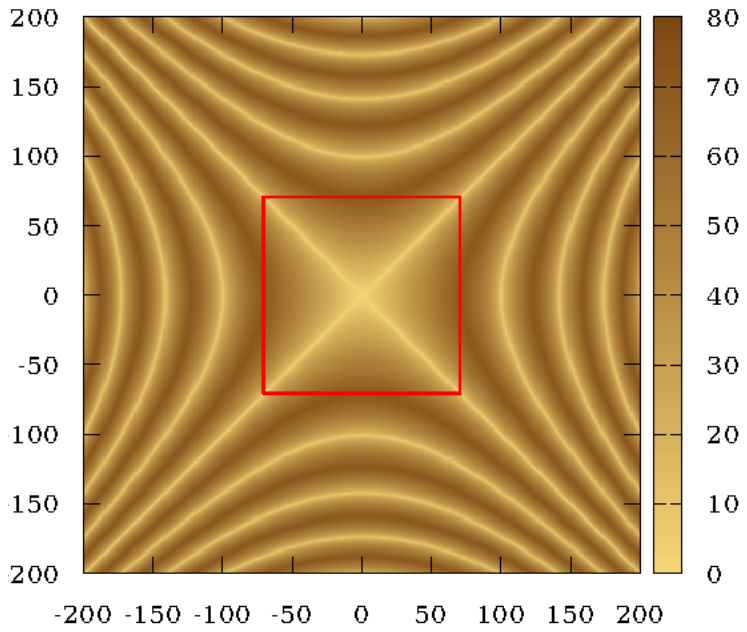


Figure 3.1: Local world domain for $q = 10007$ inside the red box for a hyperbolic biquadric. The range of the plot is restricted to show the local world domain better. Inside this domain the distances s of a point x to the center c ($s(x) = \sqrt[\mathbb{R}]{|M_{ij}x^i x^j \bmod q|}$ with $i, j \in [1, d]$) which are encoded in colors behave as we know it from our real world. Outside the local domain the periodicity of the prime field destroys the order of distances.

points outside this local domain was to refold them into the local domain. This done as shown in [Mec17] on p. 5 by calculating

$$\underline{x_\nu} = \operatorname{sgn}(x_\nu) \lfloor \sqrt[\mathbb{R}]{x_\nu x_\nu \bmod p} \rfloor. \tag{3.1}$$

To get more intuition about the properties of this local domain, the lines in the affine plane are refolded to the local domain. At the beginning of

the project this refolding was one idea to introduce quantum features into the finite world. In the meantime the refolding method shown here was replaced by another idea, where the associated biquadric points are refolded into a fiber attached to the corresponding center point in the quadratically ordered local domain. Klaus Mecke's idea is to introduce quantum features and elementary particles by the fiber construction, as it resembles the fiber bundle construction in quantum field theory more. This would result in a complete geometrical unification of space, time and matter.([Mec18], p. 39)

3.1.1 Refolding procedure - general information and construction

The points x that are incident with a line l are given as the solutions of $l^t x = 0$.
In the following we are only interested in affine points[1] and lines that contain the origin $(0,0,1)^t$. Therefore we can set $x_2 = 1$ and $l_2 = 0$ what simplifies our incidence equation above.
It is possible to manipulate the equation to see directly which condition must be fulfilled by affine points to be on the line l. One can write all these points in the following way

$$x = \begin{pmatrix} x_0 \\ \frac{-l_0}{l_1} x_0 \bmod q \\ 1 \end{pmatrix} = \begin{pmatrix} x_0 \\ m x_0 \bmod q \\ 1 \end{pmatrix}. \tag{3.2}$$

The division here is done in the finite field, so m is an element of the finite field and will be treated like an integer in the following. This $m = m_{proj}$ can be interpreted as the "projective slope" of a line, in contrast to $m_{\mathbb{R}}$ which we will call the "real" slope. Here the division would be done in the real numbers. In the following the projective slope will be simply called m, when we speak of the "real" slope this will be made clear.
This explicit form of a line can now be used to do the refolding, which is

[1] We choose $(0,0,1)$ as line at infinity and $(0,0,1)$ as the origin.

given by the following rule ([Mec17], p. 6)

$$\underline{x_\nu} = \mathrm{sgn}(x_\nu)\lfloor\sqrt{x_\nu x_\nu \bmod q}\rfloor. \tag{3.3}$$

Applying this rule to the parametrized points of the line as given in equation 3.2 we get the refolded line in the local world domain.
Clearly visible are the different branches that occur due to the refolding and the periodicity of the whole geometry. Doing the refolding step by step helps to understand why the refolded line looks as it does. In a first step one only looks at the square of the points on the line. This is given by

$$\begin{aligned} x^2 &= \begin{pmatrix} x_0 \cdot x_0 \bmod q \\ \frac{(-l_0)^2}{l_1^2} x_0 \cdot x_0 \bmod q \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} x_0 \cdot x_0 - \alpha(x_0) \cdot q \\ m^2 x_0 \cdot x_0 - \beta(x_0) \cdot q \\ 1 \end{pmatrix}. \end{aligned}$$

Where after the second equal sign we replace, for an easier analysis, the modulo-operation by some functions $\alpha(x_0)$ and $\beta(x_0)$ which are given by

$$\alpha(x_0) = \left\lfloor \frac{x_0 \cdot x_0}{q} \right\rfloor \quad \text{and} \tag{3.4}$$

$$\beta(x_0) = \left\lfloor \frac{m^2 x_0 \cdot x_0}{q} \right\rfloor. \tag{3.5}$$

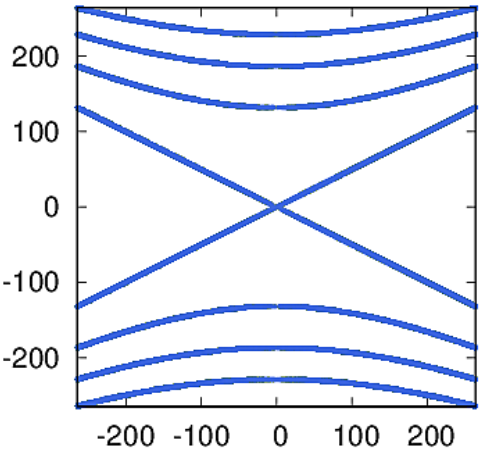
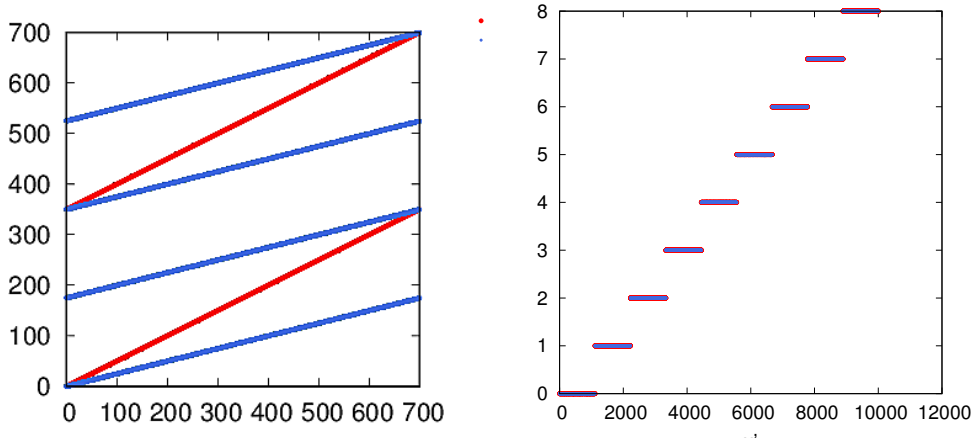


Figure 3.2: Refolded line (1, 2, 0) in the local domain for $q = 70001$.

Now we do not apply the modulo-operation anymore, all of this is now done by the $\alpha(x_0)$ an $\beta(x_0)$-functions. An interesting question now is, which form does the square of a line have in a projective space? Plotting the result given above results again in a line through the origin, but with a squared slope compared to the original line. To see this also from the calculation above, one defines $x' = x_0 \cdot x_0 - \alpha(x_0) \cdot q$. Inserting this, we



(a) Original line (1, 2, 0) in blue and squared line (1, 4, 0) in red for $q = 70001$, both axis scaled by factor 0,01.

(b) Equivalence of $\gamma(x')$ in red and $-\alpha(x_0)m^2 + \beta(x_0)$ in blue, x-axis in terms of x', $q = 10007$, line $l = (-3, 1, 0)$.

Figure 3.3: Illustration of equ. 3.6, showing that squaring maps projective lines onto projective lines with squared slope.

obtain

$$x^2 = \begin{pmatrix} x' \\ m^2x' - (-\alpha(x_0)m^2 + \beta(x_0)) \cdot p \\ 1 \end{pmatrix}. \tag{3.6}$$

This already looks a lot like the equation of a projective line.
But if we want to see that clearly, we have to introduce a new function $\gamma(x_0) = \left\lfloor \frac{m^2 \cdot x_0}{q} \right\rfloor$. This is the function that replaces the modulo operation in a normal line equation with a slope equal to m^2. Now one has to show that $\gamma(x') = -\alpha(x_0)m^2 + \beta(x_0)$. Numerical simulations as shown in fig. 3.3(b) make the equivalence plausible, but it can also be shown analytically. This

is done in the following.

$$\begin{aligned}\gamma(x') &= \left\lfloor \frac{m^2 \cdot x'}{q} \right\rfloor = \\ &= \left\lfloor \frac{m^2(x_0 x_0 - \alpha(x_0)p)}{q} \right\rfloor = \\ &= \left\lfloor \frac{m^2 x_0 x_0}{q} - m^2\alpha(x_0) \right\rfloor = \\ &= \left\lfloor \frac{m^2 x_0 x_0}{q} \right\rfloor - m^2\alpha(x_0) = \\ &= \beta(x_0) - m^2\alpha(x_0).\end{aligned}$$

Where we can pull $m^2\alpha(x_0)$ out of the floor-bracket in the third step, because it can only take integer values and therefore does not affect the rounded result given by the floor-function.

This calculation can be done for arbitrary powers, not just squares. Therefore we see, that taking the line to some arbitrary power results again in a line, but with a slope equal to the original slope taken to the same power as the whole line. This is the point where it makes sense to exclude lines that do not contain the origin, because taking such lines to any arbitrary power does not result in another line with a different slope. Very much in contrast: The new, for example squared points are now uniformly distributed in the whole projective plane. Now we can go on and investigate the role of the square root in the refolding rule. The square root is taken in the real numbers,

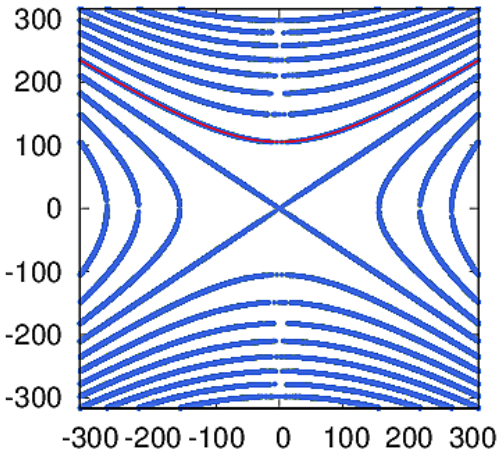


Figure 3.4: Refolded line (2, 3, 0) in blue, one upper branch fitted by $\sqrt{ax^2 + b}$, plotted in red.

not in the Galois field underlying our projective geometry. If one would take the square root in the Galois field, the refolded line would simply be exactly the same as the original line we started with. By taking the real square root, one obtains a result in the local domain from $-\lfloor\sqrt{q}\rfloor$ to $\lfloor\sqrt{q}\rfloor$. Doing the refolding now completely gives

$$x = \begin{pmatrix} \mathrm{sgn}(x_0)\sqrt{x'} \\ \mathrm{sgn}(x_1)\sqrt{m^2x' - \gamma(x') \cdot q} \\ 1 \end{pmatrix} = \\ = \begin{pmatrix} \mathrm{sgn}(x_0)x'' \\ \mathrm{sgn}(x_1)\sqrt{m^2x''^2 - \gamma(x') \cdot q} \\ 1 \end{pmatrix}.$$

Where we defined once more a new parameter $x'' = \sqrt{x'}$. From this expression one can now read off how the refolded line looks. It has several branches due to the term $\gamma(x')q$ that is added at the end of the expression that can take several values depending on the chosen x'.
In the following we will also calculate how many branches the refolding will have. But at first one can analyze whether the $\sqrt{ax^2 + b}$-dependency is in fact in accordance with what simulations give. For the prime number $q = 99997$ and the original line $l = (2, 3, 0)$ one would expect a "real" slope of $m_{\mathbb{R}} = \frac{-2}{3}$, therefore $a = m_{\mathbb{R}}^2 = \frac{4}{9} \approx 0,444....$ The branch one chooses for the fitting determines the value of b. Doing this fitting and plotting everything together results in the fig. 3.4.
One can see that the agreement of fit and calculation is very good (fitted value of $a = 0,4416 \pm 0,0005$ and $b = 11105 \pm 140$). Therefore we conclude that indeed the refolded line can be approximated by square-root functions. What makes this functions interesting is its limit behavior, if we go to very big values for x'' the square root converges towards the original line again.

3.1.2 Number of branches

Another question one can pose is: How and why do the different branches of the refolding occur? How many are there? This shall be answered now. Therefore it is important to remember that we work in a projective space whose properties are markedly different from the well-known real space. But during the refolding only the square is taken in the projective space,

the square root is taken in the real numbers. Therefore it is not the inverse operation of the square in this case. One notices that already after taking the square, the line has several branches. As the square root here is not the inverse of the square it does not affect the number of branches, but only transforms the form of the branches that are already there. The square root sees all of these branches as real lines with the same slope that differ only by their y-axes intercept. To compute the number of branches we interpret the projective line we get as many real lines that differ by their axis intercept. This axis intercept can be computed and the number of solutions of the occurring equation defines the number of different intercepts and therefore different branches.
Here a general approach is possible because the square of a line is again a line with squared slope as it was shown above. This is the reason why we can do everything in the following for ordinary lines which also have different "real" branches. Later this can be easily adapted to the squared lines that appear during the refolding process.
For a more careful analysis one recognizes that the slope of a line in the projective space and the real numbers is different. As we want the slope in the real numbers, we start by segmenting the projective slope into a real and a projective one.

$$\begin{aligned} x_1 &= m_{proj}x_0 - \epsilon(x_0)q = \\ &= m_{\mathbb{R}}x_0 + \\ &+ (m_{proj} - m_{\mathbb{R}})x_0 - \\ &- \epsilon(x_0)q \end{aligned}$$

where $\epsilon(x_0)$ again replaces the modulo operation and is given by $\epsilon(x_0) = \left\lfloor \frac{m \cdot x_0}{q} \right\rfloor$. Now it is clear that the different y-axis intercepts of

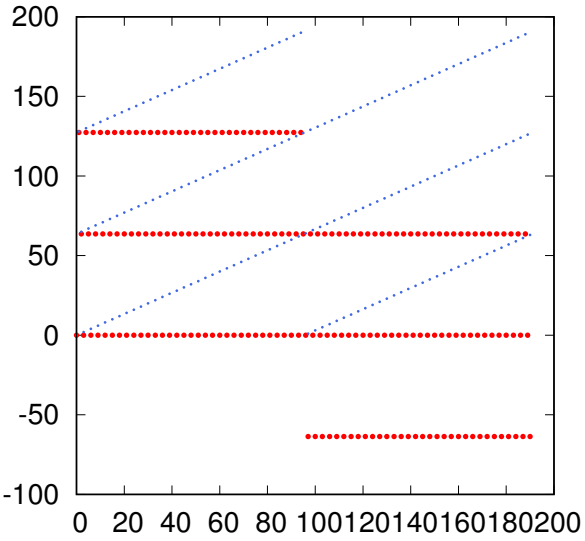


Figure 3.5: Different branches of a line in blue and y-axis intercepts in red for $l = (-2, 3, 0)$ and $q = 191$.

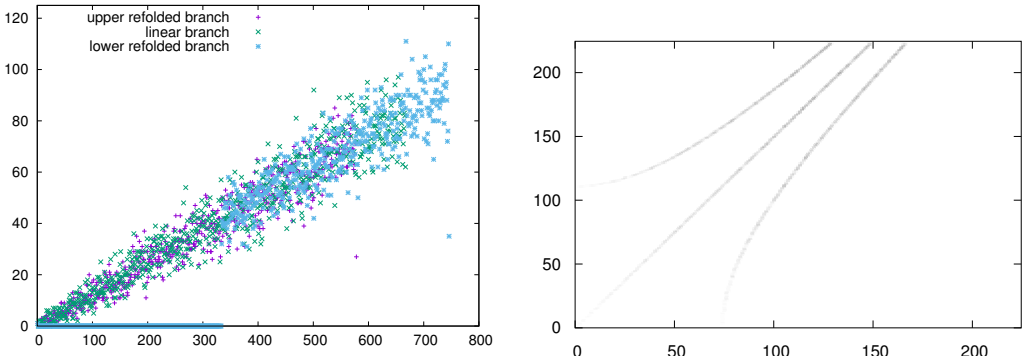
the branches, now interpreted as real lines, are given by the different values $(m_{proj} - m_{\mathbb{R}})x_0 - \epsilon(x_0)p$ can take for all x_0 from the finite field.
Plotting an example for $l = (-2, 3, 0)$ shows that every x_0 has only one intercept and therefore only lies on one real line. Here the word "real" is maybe misleading as it conveys the impression that the lines are continuous what is not true as all the lines only consist of discrete points. "Real" here just refers to the slope the corresponding line would have in the real numbers.
What would be nice to have is a general procedure of how to compute how many different axis intercepts can occur, depending on the investigated line $l = (l_0, l_1, 0)^t$. Up to now such a procedure is not known, from simulations it seems like there are always $|l_0| + |l_1| - 1$ many such branches of which $|l_0| - 1$ have negative and $|l_1| - 1$ have positive axis intercept. For the lines that appear during the refolding l_0 and l_1 are squared. Therefore the number of positive branches gets $l_1^2 - 1$ and the number of negative branches $l_0^2 - 1$ what coincides with the number of refolded lines in the different quadrants in the simulations shown in the examples above. The symmetric structure of the refolding is highly influenced by using the sign $\mathrm{sgn}(x_0)$ and $\mathrm{sgn}(x_1)$ of the original line-points at the end of the refolding assignment.

3.1.3 Distribution of refolded points in the local domain

During the refolding procedure different points of the projective space are mapped to the same point in the local domain. One can now look at the number of points that are mapped to the same local point and investigate how this depends on the original line that is refolded.
To make this plausible one notices that by the definition of our refolding for all $x \in \mathbb{L}$ all points in the interval $[x^2, (x+1)^2[$ are mapped to x and $(x+1)^2$ is excluded from the interval. The number of points in this interval increases linearly, which can be seen quickly. Even the slope m of the linear increase can be read off.

$$\#N_{interval} = (x+1)^2 - x^2 = 2x \rightarrow m = 2$$

This is of course only true if all points in the interval $[x^2, (x+1)^2[$ are occupied, what is not true for the lines we look at. Nevertheless, one can examine if the linear behavior is reproducible in this case.
This is done by generating a heat map of the refolded line and then counting for each value on the x-axis the number of points that is mapped to this x-value. The obtained data can be used to generate a histogram, which tells how the amount of points refolded to the same point changes in terms of the corresponding x-value. The example in fig. 3.6 shows the whole procedure for $l = (3, -2, 0)^t$. The y-axis of the histogram shows the number of refolded points for the corresponding x-value in the local domain. The



(a) Points in each branch for $q = 1000007$ (b) Chosen branches for the histogram for $q = 49999$

Figure 3.6: Histogram for three different refolded branches. All three have the same slope.

histogram is shown for $q = 1000007$, because at such large scale the linearity of the plot is easy to see, whereas the heat map is shown for $q = 49999$ as smaller primes allow for a better visualization of the refolded lines in the local domain.
One clearly sees that the number of points increases linearly, independently of the chosen branch. The linear increase is disturbed, because not in every interval $[x^2, (x+1)^2[$ corresponding to some value x in the local domain the same percentage of points is occupied, therefore sometimes the number of refolded points is higher than expected and sometimes also smaller.
The slope of $m = 2$, which would be expected if every point was occupied, could not be reproduced, as already mentioned above, but nevertheless it

would be interesting to find an analytic expression for the slope depending on the parametrization of the refolded line. This is done in the following. As we still concentrate on lines through the origin, the parametrization is given up to integer multiples which we do not look at due to homogeneity by $l = (l_0, l_1, 0)$, and after the squaring which occurs during the refolding $l^2 = (l_0^2, l_1^2, 0)$ to whose components we will refer with l_i' in the following, where $i \in 0, 1$.
The idea now works as follows: Throughout the whole calculation we treat the projective line like several real lines in euclidean space. First we calculate the length of shortest occurring piece (in the following: elementary piece) of the line by using the Pythagorean theorem, then we calculate the length of the whole line if we glue all the occurring branches together and calculate the point density. Our last step is to detect of how many elementary pieces the branch we currently look at exists and to calculate from this the slope of the histogram. Here we do the calculation for simplicity only for those branches that start at $x = 0$. The following figure shows the important pieces of the calculation in the plot.
The length l_{elem} of the elementary line-piece is given due to the Pythagorean theorem by

$$l_{elem} = \sqrt{\left(\frac{q}{l_0'}\right)^2 + \left(\frac{q}{l_1'}\right)^2} = q\sqrt{\frac{1}{l_0'^2 + l_1'^2}}.$$

Analogously one can calculate the length of the longest piece in the real numbers:

$$l_{long} = \sqrt{q^2 + \left(\frac{l_1'}{l_0'}q\right)^2}.$$

By gluing together all the branches one can build this longest piece exactly l_0' times, therefore the whole length of the line expressed in the real numbers is $l_{line} = l_0' \cdot l_{long}$.
With this knowledge it is possible to calculate the point density on the line. Here our first assumption enters the derivation: We assume that the points are distributed uniformly along the line, such that in each interval $[x^2, (x+1)^2[$ the same number of points is occupied. Throughout this paragraph we concentrate on the upper right quadrant of the affine plane,

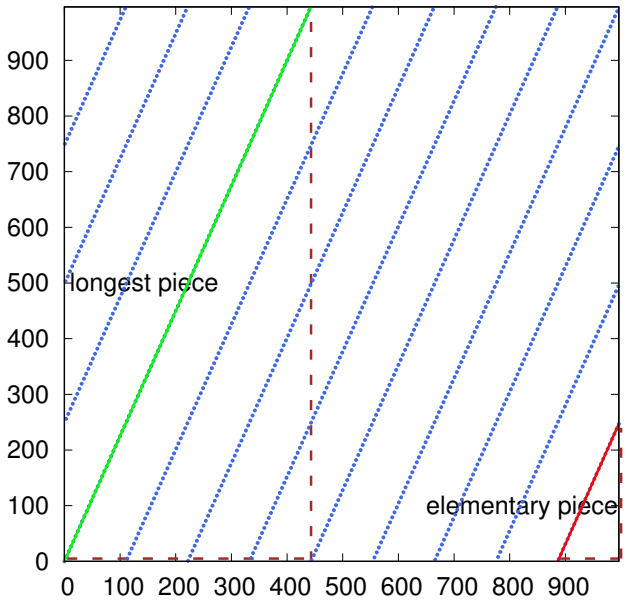


Figure 3.7: Important pieces of the line for $q = 997$ and $l = (3, 2, 0)$ after squaring. Elementary piece in red and longest piece in green.

thats why we only have $\frac{q}{4}$ points on the line[2]. The point-density is

$$\rho_{point} = \frac{\frac{q}{4}}{l'_0 q\sqrt{1 + (\frac{l'_1}{l'_0})^2}} = \\ = \frac{1}{4\sqrt{l'^2_0 + l'^2_1}}.$$

To calculate the slope of the histogram one needs the number of points on the investigated branch, which is now easy to compute. Suppose the

[2]In the whole projective plane, consisting of 4 quadrants, $(q + 1)$ points are incident with the line.

branch consist of x elementary pieces, then the number of points is

$$N_{branch} = x l_{elem} \rho_{point} = \frac{x \cdot q}{4} \frac{\sqrt{\frac{1}{l_0'^2 + l_1'^2}}}{\sqrt{l_0'^2 + l_1'^2}} = \\ = \frac{x \cdot q}{4 l_0' l_1'}.$$

Now the next assumption we do for the derivation of the slope is, that the histogram follows a linear increase. Then we get

$$m = \frac{\Delta N_{branch}}{\Delta x} = \frac{\frac{\text{number of points on the branch}}{\text{length in local domain}} \cdot 2}{\text{length in local domain}}.$$

Where Δx is the length of the histogram in the local domain and ΔN_{branch} is how we distribute the points on the branch in the local domain. With these assignments the slope is quickly computed.

$$m = \frac{\frac{xq}{2 l_0' l_1'}}{\sqrt{\frac{x}{l_0'} q} \sqrt{\frac{x}{l_0'} q}} = \tag{3.7}$$

$$= \frac{1}{2 l_1'} \tag{3.8}$$

Therefore the slope of the histogram only depends on the value of l_1. Integer multiples of the line l have the same slope. The calculation for branches that do not start at $x = 0$ is longer but leads to the same result.

4 Investigation of biquadric fields

> Noli turbare circulos meos! (*Do not disturb my circles!*)
>
> Archimedes (no evidence)

After the detailed introduction of biquadrics and their transformation behavior, as well as the different types and the notion of a biquadric field, it is time to face some properties of these objects. In this chapter of the thesis we will mainly concentrate on the search for *flat* biquadric fields $\mathcal{B}$ in the projective plane, as these would be interesting for different applications.
Furthermore we will deal with the question whether the point set of a biquadric determines the representation matrices uniquely[1] and if transformations $\Lambda \in \mathrm{Aut}(\mathbb{P}^d\mathbb{F}_q)$ exist, that keep the point set invariant but change the representation matrices. These transformations would be an unknown part of the *Lorentz-group* which is defined as the group of all automorphisms that keep the biquadric invariant.

4.1 Search for flat spacetime

In General Relativity curvature is a key notion for the description of space-time and determines the movement of particles through space-time. For the finite projective geometry there is no such notion as curvature defined

[1] It could be that different representation matrices exist, that have the same point set.

by now, but as everything is countable due to the finiteness, one idea is to count for each point how often it is contained in the neighborhood (i.e. the biquadric) of all other points.

- If all points are equally often contained in the neighborhood of all other points, the biquadric field is called *flat* or *homogeneous*.
- If the countrate of several points differs from the other points one says the spacetime is *curved* and the biquadric field *inhomogeneous*.

By now, it was not defined precisely what it means in terms of curvature, if a point is neighbor of more (or less) points than the average countrate. The average countrate is well-defined if at all points in the spacetime a biquadric of the same type is attached - then for elliptic biquadrics each point has $2(q+1)$ neighbors and is neighbor of $2(q+1)$ points in average. For the hyperbolic case each point has $2q$ neighbors and is contained in $2q$ neighborhoods in average.
To avoid confusion it is important to note that this chapter concentrates entirely on the search for flat biquadric fields within the whole projective plane, including all q^2+q+1 points. It would also be possible to search only for flat fields in the affine plane, but here it is easy to give such a field by affine translations - this will be used in upcoming chapters.
In his Master's Thesis Alexander Laska ([Las14]) found a flat biquadric field for $\mathbb{P}^2\mathbb{F}_3$ in the whole projective plane, but only using biquadrics that have more than one center point. As we restrict to biquadrics with unique center, it would be very interesting to find a flat field using just unique biquadrics. Therefore an algorithm has been written and applied to different prime fields.

4.1.1 Symmetry condition

Before the algorithm is presented, we take a step back and think about what symmetries could be reasonably demanded of physical biquadric fields.
One approach followed by Klaus Mecke to implement symmetry is the so-called *symmetry condition*([Mec17], p. 6 and [Mec18], p. 25).

Definition 4.1.1. Symmetry condition A biquadric field $\mathcal{B} : \mathbb{P}^d\mathbb{F}_q \mapsto \mathfrak{B}$ is said to obey the symmetry condition if

$$\forall\, a, b \in \mathbb{P}^d\mathbb{F}_q : b \in \mathcal{B}(a) \Leftrightarrow a \in \mathcal{B}(b). \tag{4.1}$$

This is just the mathematical way of demanding, if one point is the neighbor of another point, then the second point should also be neighbor of the first one. This condition is indeed very strong, because if one would find a field that obeys the condition, the field is necessarily flat as the following theorem states.

Theorem 4.1.1. *If $\mathcal{B}$ is a biquadric field that obeys the symmetry condition, then $\mathcal{B}$ must be necessarily flat.*

Proof. This theorem will be proven by contradiction. Suppose $\mathcal{B}$ is a biquadric field that is not flat but obeys the symmetry condition.
For such a field at least one point exists that is in more or less than $2(q+1)$(elliptic) respectively $2q$ (hyperbolic) many neighborhoods. But this point can only have precisely $2(q+1)$(elliptic) respectively $2q$ (hyperbolic) many neighbors, therefore two cases have to be considered now:

- If the point is in less than $2(q+1)$(elliptic) respectively $2q$ (hyperbolic) neighborhoods, another point must exist that is in more than $2(q+1)$(elliptic) respectively $2q$ (hyperbolic) neighborhoods, because the overall countrate is fixed by the fact that every point has a fixed number of neighbors. This point that is in more neighborhoods than it has neighbors has to fulfill the symmetry condition with respect to all points of which it is neighbor because the biquadric field as a whole obeys the symmetry condition. This is not possible as the point cannot have more than $2(q+1)$(elliptic) respectively $2q$ (hyperbolic) neighbors and therefore the field cannot obey the symmetry condition.

- If a point is in more than $2(q+1)$(elliptic) respectively $2q$ (hyperbolic) neighborhoods the same argumentation as above can be applied and leads again to a contradiction.

Both cases lead to contradictions, therefore we have proven that a biquadric field that obeys the symmetry condition has to be flat. □

The theorem says nothing about the existence of symmetric or flat fields, it just states that if a symmetric field exists, it is necessarily flat. *The other direction (flat fields have to be symmetric) of this theorem is indeed not true*, as already flat fields have been found that are not symmetric. One

of these fields is explicitely presented later for the case $\mathbb{P}^2\mathbb{F}_3$ - it is straight forward to check that this field does not obey the symmetry condition.
If we just concentrate on the affine plane as our physical space it is straightforward to construct flat and symmetric fields by translating biquadrics in the affine plane. As translations are automorphisms of the affine space, they preserve incidence. As a reminder, the translation matrix from a given point $p \in \mathbb{P}^d\mathbb{F}_q \setminus l_\infty$ to $q = (p + t) \in \mathbb{P}^d\mathbb{F}_q \setminus l_\infty$ is given by

$$T_{\vec{t}} = \begin{pmatrix} \mathbb{1}_{d\times d} & \vec{t} \\ \vec{0}^t & 1 \end{pmatrix}.$$

A flat and symmetric translational biquadric field in the affine plane is now given by translating a given biquadric from the standard center point $(0, ..., 0, 1)^t$ to each other affine point.[2] These affine translations will never change the biquadric points that are incident with the line at infinity, because they cannot be translated by the matrix given above.[3] Therefore the emerging biquadric field is only flat in the affine space. On the line at infinity in the case of hyperbolic biquadrics 2 points would be contained in every single biquadric, and the rest in no biquadric at all, therefore this would destroy the flat state. For the elliptic case, no biquadric at all would contain a point at infinity, destroying the projectively flat state as well. Nevertheless this affine flatness is very important for later considerations in the *Game of Life* and therefore already introduced here.

[2] This is the case because biquadric fields are point symmetric with respect to their center point, and if such a biquadric is translated to one of its biquadric points the former center point will for sure be a biquadric point of the resulting biquadric and the two fulfill the symmetry condition.

[3]

$$\begin{pmatrix} \mathbb{1}_{d\times d} & \vec{t} \\ \vec{0}^t & 1 \end{pmatrix} \cdot \begin{pmatrix} \vec{p} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbb{1}_{d\times d} \cdot \vec{p} + 0 \cdot \vec{t} \\ \vec{0}^p + 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} \vec{p} \\ 0 \end{pmatrix}$$

where $\vec{p}$ denotes the components of the point at infinity.

4.1.2 Attempts for different prime numbers

As projective geometries do not only consist of the affine part but also of the hyperplanes at infinity it would be very interesting to have a general construction manual for flat biquadric fields in the whole projective space. In his Master's thesis Alexander Laska already tried to construct such fields by transforming the biquadrics in specific ways ([Las14], p. 80) without finding a flat state. Only for $\mathbb{P}^2\mathbb{F}_3$ an explicit flat state was constructed by using biquadrics that have non-unique center points ([Las14], p. 86).
In this theses two other approaches are taken. The first is to consider all possible biquadrics with unique center and polar and distribute them in the projective space such that the resulting field is flat. The second approach transforms a given biquadric to all possible center points with all possible projectivities.

Approach 1: Placing biquadrics in the projective space

The idea is to calculate all uniquely centered biquadrics, place them at their center point one after another and calculate the countrate of the points in the geometry until all points have them same countrate. But before this can be done the question is: How does one prepare the biquadrics for the algorithm?
Firstly, all symmetric, non-degenerate, homogeneous 2×2 submatrices m are calculated for a given prime q and stored in a vector. For each of these matrices one hyperplane at infinity h_∞ and two hyperplanes h_1, h_2 through the desired center point are chosen. Furthermore one square and one non-square are chosen to construct the corresponding representation matrices as shown in equ. 2.5.[4]. In this fashion for each possible submatrix all combinations are taken. As one can take several combinations of h_1 and h_2 that still yield p_c as center point, not all of the resulting biquadrics are unique. Therefore all of those that occur more than once are taken out until only one of each biquadric is left. Like that all biquadrics with unique center point and unique polar that can be generated from the application of equ. 2.5 are constructed. By now there is no proof that the method

[4]The center points are calculated according to $h_1 \times h_2 = p_c$.

in equ. 2.5 generates all possible uniquely centered biquadrics. It could be that more unique biquadrics exist which cannot be constructed in the shown manner.
It is not surprising that the number of constructed biquadrics increases strongly with the chosen prime field. For $q = 3$ one finds 1053 different biquadrics[5], 81 for each center point and 9 for each combination of center point and hyperplane at infinity. From these 9 only 3 are elliptic and 6 are hyperbolic. For higher prime numbers the amount of such biquadrics increases like

$$N_{biquadrics} = q^2 \cdot q^2 \cdot (q^2 + q + 1) \cdot \frac{q-1}{2} \cdot \frac{q-1}{2} \in \mathcal{O}(q^8).$$

For $q = 5$ one can find 77500 biquadrics in total, 2500 per centerpoint and 100 for each combination of centerpoint and hyperplane. 40 of these are elliptic and 60 hyperbolic. If one wants to test all possible elliptic biquadric fields for $q = 5$ there would be $40^{31} = 4.61 \times 10^{49}$ possiblities what is simply not possible to handle in decent time with the computer. Therefore clever reduction of biquadrics is needed. This will be explained later.
The algorithm now starts by placing a random biquadric on its centerpoint. This is done for one point after another, until either all points have been reached and the field is flat, or until the countrate of one points exceeds the critical value $2(q + 1)$ or $2q$.
If this happens the last chosen biquadric is removed and another one is taken for the same centerpoint. If no suitable biquadric[6] in the momentary step can fulfill the flat field requirement anymore, the penultimately chosen biquadric is removed as well and another suitable biquadric for the this step is searched. This is done iteratively until either all possible biquadric combinations have been tested or until a flat field is found. If all biquadric combinations are tested, no flat field exists - for higher prime numbers than 3 this takes an enormous amount of time as one can guess from the high number of possible combinations given above. If the algorithm finds a flat

[5] 9 submatrices, 9 different hyperplanes at infinity for each center point, and 13 different center points, 1 square and 1 non-square: $9 \cdot 9 \cdot 13 \cdot 1 \cdot 1 = 1053$.
[6] 'Suitable' means the biquadric has to have the centerpoint for which a biquadric is searched at the moment and the countrate does not exceed the critical value.

state, it prints the biquadrics, the center points and the hyperplanes at infinity to a file. This data can be used to find a general mechanism of how to construct flat biquadric fields.
For $\mathbb{P}^2\mathbb{F}_3$ hyperbolic and elliptic flat fields both exist. An explicit example for a flat biquadric field is given by

p_c	l_∞	$(M, \overline{M})$	$\mathcal{B}_{M,\overline{M}}$	h_1, h_2
$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}2\\2\\1\end{pmatrix}$	$\begin{pmatrix}2&2&1\\2&0&0\\1&0&1\end{pmatrix}, \begin{pmatrix}1&1&2\\1&2&1\\2&1&0\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$
$\begin{pmatrix}1\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\2\\1\end{pmatrix}$	$\begin{pmatrix}2&1&1\\1&2&1\\1&1&0\end{pmatrix}, \begin{pmatrix}2&1&1\\1&0&0\\1&0&1\end{pmatrix}$	$\begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}$
$\begin{pmatrix}2\\0\\1\end{pmatrix}$	$\begin{pmatrix}2\\0\\1\end{pmatrix}$	$\begin{pmatrix}0&1&1\\1&1&1\\1&1&0\end{pmatrix}, \begin{pmatrix}1&1&0\\1&1&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}0\\2\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}$
$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}1&2&2\\2&0&0\\2&0&1\end{pmatrix}, \begin{pmatrix}2&1&1\\1&1&1\\1&1&0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}$
$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0&2&0\\2&2&2\\0&2&1\end{pmatrix}, \begin{pmatrix}2&2&0\\2&2&2\\0&2&1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}$	$\begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}$
$\begin{pmatrix}2\\2\\1\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1&2&0\\2&2&1\\0&1&0\end{pmatrix}, \begin{pmatrix}2&1&0\\1&1&2\\0&2&1\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}$
$\begin{pmatrix}2\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\2\\1\end{pmatrix}$	$\begin{pmatrix}0&2&1\\2&0&0\\1&0&1\end{pmatrix}, \begin{pmatrix}2&0&0\\0&2&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}0\\2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}$
$\begin{pmatrix}1\\2\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&1&2\\0&2&1\end{pmatrix}, \begin{pmatrix}2&0&2\\0&0&1\\2&1&0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}$
$\begin{pmatrix}2\\1\\1\end{pmatrix}$	$\begin{pmatrix}2\\1\\0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&2&0\\1&0&1\end{pmatrix}, \begin{pmatrix}1&2&1\\2&0&0\\1&0&1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\2\\1\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}$

p_c	l_∞	$(M, \overline{M})$	$\mathcal{B}_{M,\overline{M}}$	h_1, h_2
$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\1&1&0\\1&0&1\end{pmatrix}, \begin{pmatrix}1&2&1\\2&2&0\\1&0&1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\2\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}$
$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$	$\begin{pmatrix}0&2&0\\2&2&1\\0&1&1\end{pmatrix}, \begin{pmatrix}2&2&2\\2&2&1\\2&1&0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}0\\2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}$
$\begin{pmatrix}0\\2\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}2&1&1\\1&2&1\\1&1&1\end{pmatrix}, \begin{pmatrix}2&1&1\\1&0&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}$
$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}2\\1\\1\end{pmatrix}$	$\begin{pmatrix}0&0&2\\0&2&1\\2&1&1\end{pmatrix}, \begin{pmatrix}2&1&2\\1&0&1\\2&1&1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}$

Table 4.1: Uniquely centered flat biquadric field for $\mathbb{P}^2\mathbb{F}_3$ found by suitably assigning biquadrics to center points.

The small submatrix used for the construction was $\mathrm{m} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$. Due to the choice of hyperplanes h_1, h_2 with $h_1 \times h_2 = p_c$ the explicit form of the biquadric can vary - it is not clear yet, if every possible set of combinations for h_1 and h_2 gives a flat field. For higher prime numbers one has to choose also the square q and non-square $\bar{q}$ for the construction, for $q = 3$ there is only one choice for each of them.
So far all flat fields found for $q = 3$ violate the symmetry condition given in the chapter above in the sense that not every point is its neighbors neighbor. As symmetric fields necessarily have to be flat, this leads to the question whether such symmetric fields even exist if at every point of the finite geometry a biquadric is attached. One way out of this dilemma could be to leave some points "naked", such that they are not equipped with a biquadric. This idea is not further investigated during this thesis, but followed by Klaus Mecke and seen as one way to introduce quantum features into the finite projective world.
What is remarkable about the flat fields for $q = 3$ is the fact that each hyperplane at infinity is chosen once, and that the construction submatrix is always the same for all biquadrics in the field. This may lead to a new idea of how to construct such fields for higher prime numbers. Instead

of trying all possible combinations, what would take an almost impossible amount of time, a restriction to a small subset is possible if the construction mechanism is the same here. Therefore for $\mathbb{P}^2\mathbb{F}_5$ the first step was to assign to each center point $p_{c,i}$ a different hyperplane at infinity $h_{\infty,i}$. From this all biquadrics $(M_i, \overline{M_i})$, constructed from the chosen submatrix, with $\mathrm{pol}_M(p_{c,i}) = h_{\infty,i}$ were calculated. Also the hyperplanes h_1 and h_2 were taken as given here. The only possible freedom of choice one has now is in terms of the square q and the non-square $\bar{q}$. In $\mathbb{P}^2\mathbb{F}_5$ 2 squares and 2 non-squares exist, therefore 4 possible biquadrics for each center point. The number of all possible combinations is $4^{31} = 4.61 \times 10^{18}$.
For checking 2^{31} combinations the algorithm needs about 50 minutes, resulting in approximately 200000 years for 4^{31} combinations. Even if the process is started on several cores simultaneously it is still impossible to iterate over all combinations. What may help us finding a flat state faster is, that once a flat state is found, all projectivities will hold it invariant, because they preserve incidence. Therefore, if one flat state is found it must be connected to as many other flat states as projectivities exist. For $q = 5$ there are 1488000 possible projectivities, what was calculated by a short C++ algorithm. This means that after about 50 days a flat state should be found if it exists. Until now this is not the case. If the calculation with 4 possible biquadrics at each point does not find a flat state, one would have to check with all 40 biquadrics at each point - but the estimation above shows that this is nearly impossible due to an enormous amount of calculation time. The best solution would surely be to find an analytical recipe how to construct flat fields - until now this is an open question.

Approach 2: Transforming a given biquadric

A second approach to flat fields was influenced by the transformation of biquadrics. If biquadrics exist such that the corresponding field is flat, it should be possible to relate them to each other by projective transformations. First, all possible projectivities as the homogeneous, non-degenerate 3×3 matrices over $\mathbb{F}_q$ were calculated. Then one single biquadric is chosen as a starting point and transformed to all possible center points. Following *Theorem 2.70* in [Las14] on page 50, the transformation matrix Π (while simultaneously transforming the line at infinity) for points p is constructed by choosing the point p_c to which one wants to transform and

two arbitrary points p_1 and p_2 on the desired hyperplane at infinity. The dual transformation matrix is constructed by choosing two arbitrary lines which intersect the chosen new center point and a line at infinity. But as we work on the point level, the first definition is enough. The points p on the initially chosen biquadric are transformed according to

$$p' = \Pi p = \begin{pmatrix} | & | & | \\ p_1 & p_2 & p_c \\ | & | & | \end{pmatrix} p.$$

The algorithm now constructs such a transformation matrix for all possible combinations of center point and hyperplane at infinity, collects them in a vector and applies them one after another to the point set of the initial biquadric. If any value in the count map exceeds the desired value of $2(q+1)$ or $2q$ the momentary vector of transformation matrices is rejected and a new one is constructed. For $q = 3$ this results relatively fast in a flat state, whereas for higher prime numbers it seems to be difficult to even get close to a flat state. This approach has not been analyzed further, as from the $q = 3$ case no general mechanism for the transformation could be constructed, which would eventually lead to an effective reduction of possible transformations in higher order prime fields.
For $\mathbb{P}^2\mathbb{F}_3$ flat biquadric fields with unique center points were found by using transformation matrices on a given initial biquadric. For the initial biquadric

$$(M, \overline{M}) = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

defining the center point $p_c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $l_\infty = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ the following points consitute the biquadric:

$$\mathcal{B}_{(M,\overline{M})} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Tab. 4.2 shows the transformations Π that transform the initial biquadric to a new biquadric centered at $p'_c = \Pi p_c$. All of these transformed biquadric form a flat biquadric field.

Π	p'_c	$\mathcal{B}'_{(M,\overline{M})}$
$\begin{pmatrix}1&1&1\\0&1&0\\2&2&1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}$
$\begin{pmatrix}2&2&2\\0&2&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}$
$\begin{pmatrix}2&2&2\\0&2&0\\2&1&1\end{pmatrix}$	$\begin{pmatrix}2\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}$
$\begin{pmatrix}1&1&1\\0&1&1\\1&1&0\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}$
$\begin{pmatrix}1&1&1\\2&2&1\\0&1&1\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}$
$\begin{pmatrix}2&2&2\\0&2&2\\1&2&1\end{pmatrix}$	$\begin{pmatrix}2\\2\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}$
$\begin{pmatrix}1&1&1\\0&0&2\\0&1&0\end{pmatrix}$	$\begin{pmatrix}2\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}$
$\begin{pmatrix}1&1&1\\1&1&2\\0&1&1\end{pmatrix}$	$\begin{pmatrix}1\\2\\1\end{pmatrix}$	$\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}$
$\begin{pmatrix}0&0&2\\2&2&1\\0&2&1\end{pmatrix}$	$\begin{pmatrix}2\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}$
$\begin{pmatrix}2&2&0\\0&2&2\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}$
$\begin{pmatrix}1&1&0\\0&1&1\\1&0&1\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}$
$\begin{pmatrix}2&2&0\\0&2&2\\0&2&1\end{pmatrix}$	$\begin{pmatrix}0\\2\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}$

Table 4.2: Uniquely centered flat field for $\mathbb{P}^2\mathbb{F}_3$ found by projective transforamtion of an initial biquadric.

4.2 Are biquadrics unique?

Another important question that has not been answered yet, is whether the relation between the point set of a biquadric and its representation matrix is unique or whether there could be different representation matrices with the same point set. It is instructive and important to look just at simple quadrics at first. From that one can generalize to the case of biquadrics. Here an important theorem exists stating how many points define a quadric uniquely.

Theorem 4.2.1. ***Uniqueness of quadrics*** *A quadric in d-dimensional projective space is uniquely defined by*

$$N = \frac{d}{2}(d+3)$$

points, of which we already know that they can be simultaneously on a quadric.

Proof. Quadrics are represented by symmetric, homogeneous and invertible $(d+1) \times (d+1)$-matrices, where d is the projective dimension.
In general a symmetric $n \times n$-matrix has $\frac{n(n+1)}{2}$ independent entries. Due to homogeneity in projective spaces, one entry can be normalized to 1 and the number of free coefficients reduces to $\frac{n(n+1)}{2} - 1$.
Inserting now $(d+1)$ instead of n in this equation gives

$$\frac{(d+1)(d+2)}{2} - 1 = \frac{d^2+3d+2}{2} - 1 = \frac{d}{2}(d+3).$$

Thus one gets $\frac{d}{2}(d+3)$ many linear equations in the coefficients. The system of linear equations can be solved uniquely, if $\frac{d}{2}(d+3)$-many points are given, such that all equations are linearly independent.
The restriction to points from which we already know that they can be on a common quadric is necessary, because the requirements which the points have to fulfill in general is not clear by now. This will be clarified by A. Laska in his PhD-thesis, but in this thesis the case proven here is sufficient, because we are only interested in the question how many points taken from an existing quadric define that quadric uniquely. □

This means that in a two-dimensional projective space 5 points are enough to uniquely define a quadric. This can be applied to the single quadrics contained in a biquadric to get an upper limit for the number of points needed to define a biquadric uniquely. Another important theorem states that the points of a quadric define its quadratic form uniquely.

Theorem 4.2.2. ***Uniqueness of quadratic forms****([Rei16], p.15) If two quadrics $\mathcal{Q}_M$ and $\mathcal{Q}_N$ are equal as point sets, then the quadratic forms q_M, q_N describing them are equal up to a factor.*

As we work in projective spaces, where single factors are canceled by homogeneity we can set this factor without loss of generality to 1. This can now easily be combined with another theorem shown by T. Reinhart ([Rei16]) in his thesis, which states that two matrices A, B that represent the same quadratic form q are necessarily equal. Combining all of these theorems it is clear that in 2 projective dimensions 5 points are sufficient to describe a quadric and its representation matrix uniquely. One consequence is that for $q = 3$ a quadric can never be uniquely defined, as it only consists of 4 points, what makes it impossible to apply the theorems given above. For higher prime numbers than 3 at least in 2 projective dimensions every quadric and its representation matrix are unique, because there are at least 6 points (for the case $q = 5$) on it, which is enough to apply the theorems. In higher dimensions this does indeed change again.
In terms of biquadrics it is not as simple to apply the theorems above, because they consist of two quadrics. Therefore the idea was not to find an analytical solution for the problem of uniqueness, but rather to build an algorithm that searches all biquadrics with the same point set. In terms of transformations, the special ones that leave the biquadric invariant are called *Lorentz-transformations*. [Rei16] investigated these in terms of the representation matrices and the decomposition of their group structure. In this thesis we will concentrate on transformations on the point level and see if there are differences.

4.2.1 Non-unique hyperbolic biquadrics for $\mathbb{P}^2\mathbb{F}_5$

Testing non-uniqueness of representation matrices requires an algorithm that checks whether the point set of two different biquadrics is equal.

Therefore at first a random biquadric is chosen and its points are calculated, then all biquadrics for the same prime field are constructed as shown above. One after another the point set of each such biquadric is compared with the points of the initial biquadric.
For elliptic biquadrics only in the case $q = 3$ ambiguous representation matrices exist, what is a consequence of the fact that 4 points (in 2 dimensions) cannot determine a quadric uniquely. For all higher prime numbers than 3 the point set of a biquadric determines its representation matrices unambiguously. The case is quite different for hyperbolic biquadrics - here each of the two quadrics has the same points at infinity, therefore they are hit twice and removing one of the doubled points does not change the point set. But now the number of points in one of the quadrics is equal to $2(q+1)$ and in the other one $2(q-1)$. At least for low prime numbers this can make a difference. And indeed for $q = 5$ the hyperbolic case is not unique, what means that for a given biquadric always three different pairs of representation matrices exist that have the same point set. Furthermore these new biquadrics determine new centerpoints and new hyperplanes at infinity, what means that each of these biquadrics distinguishes different points at infinity.

Example 4.2.1. Example for $q = 5$: Starting with the standard matrix pair

$$(M_1, \overline{M_1}) = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

we get a biquadric with the point set[7]

$$\mathcal{B}_{(M_1,\overline{M_1})} = \left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right.$$
$$\left. \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}$$

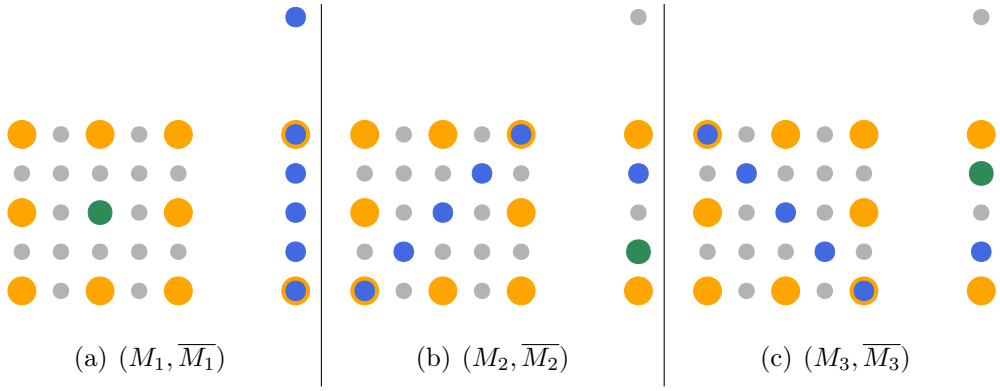
[7] Double points at infinity already removed.

with the center point $p_{c,1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $h_{\infty,1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Indeed one finds the matrix pairs $(M_2, \overline{M_2})$ and $(M_3, \overline{M_3})$ with the same point set given by

$$(M_2, \overline{M_2}) = \left(\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

$$(M_3, \overline{M_3}) = \left(\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

with $p_{c,2} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $h_\infty, 2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $p_{c,3} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $h_\infty, 3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$. The situation is displayed in figure 4.1. Furthermore from the representation



(a) $(M_1, \overline{M_1})$ (b) $(M_2, \overline{M_2})$ (c) $(M_3, \overline{M_3})$

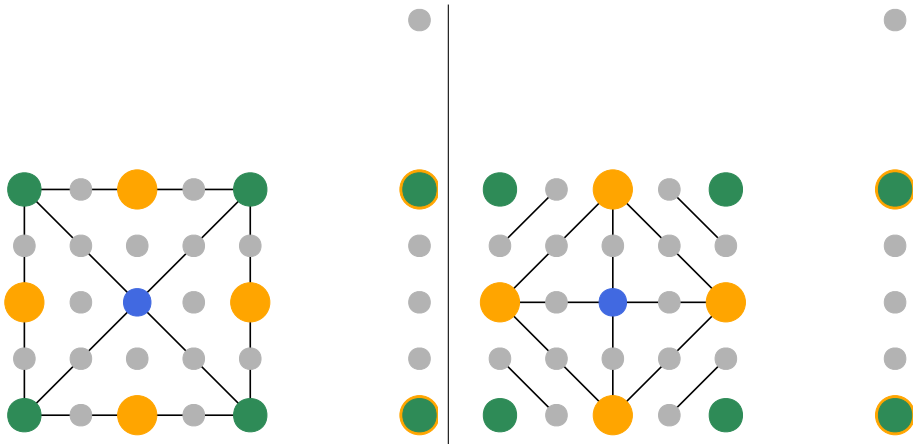
Figure 4.1: Different biquadrics (yellow) with the same point set but different representation matrices, different centerpoints (green) and different hyperplanes at infinity (blue). Remarkable is that the intersection of two hyperplanes at infinity is always equal to the remaining center point.

matrices it is obvious that only one matrix changes and the other one is invariant. Indeed this was the case for every tested biquadric so far. What happens is that a new hyperplane at infinity is chosen by removing the doubled points on the old line at infinity from one of the two quadrics.

Then one chooses one point from the other quadric and places it inside the reduced quadric - this point will determine the new line at infinity while at the same time preserving the biquadric as a point set.[8] This can only be done in 2 different ways. For example if the point $p = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ is removed and instead the point $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ (case 2) or $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ (case 3) is inserted into the second quadric then the point $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ has to be also removed and $\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$ (case 2) or $\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$ have to be chosen as a new points at infinity. These new double-counted points define the new line at infinity. This still leaves the question open, why the new infinity points can only be taken from one quadric. A graphical solution for this is given in fig. 4.2, taking into account that no 3 points of a quadric can be collinear. If one chooses to remove the infinity points from the green quadric, this quadric would have two points less than it should have. Now in order to keep the biquadric invariant the only possible choice is to take two yellow points and insert them into the green quadric - this is not possible, as all yellow points are incident with the connection lines of the green points. Therefore after assigning yellow points to the green quadric necessarily three points would be collinear. On the other hand, if the points at infinity are removed from the yellow quadric, one has two possibilities to insert green points into the yellow quadric, as these do not lie on the connection of yellow points. This makes plausible why three different representation matrices occur.

The question arises whether it is possible to calculate the representation matrices if $p_{c,i}$ and $h_{\infty,i}$ are given. Indeed this is possible if one guesses the 2×2 submatrix m, the used square q and non-square $\bar{q}$ correctly. If the

[8] Choosing one point is sufficient, because of the point symmetry of quadrics the second point is then already chosen.



(a) Lines connecting the points of M_1 (b) Lines connecting the points of $\overline{M_1}$

Figure 4.2: Drawing lines which connect the points of quadrics, one can see where it is possible to put the infinity points to generate a new biquadric. Only the affine plane is considered, as we want to put the points at infinity into this plane.

point set is given, the new hyperplanes at infinity can be constructed by choosing appropriate quadric points and replacing the points at infinity as shown above. The center points are given as the pairwise intersection of the constructed lines at infinity.

$$\begin{aligned} M_i &= q \begin{pmatrix} | & | \\ h_{\infty,(i+1)} & h_{\infty,(i-1)} \\ | & | \end{pmatrix} m \begin{pmatrix} — & h_{\infty,(i+1)} & — \\ — & h_{\infty,(i-1)} & — \end{pmatrix} + h_{\infty,i}^t h_{\infty,i} \\ \overline{M_i} &= \bar{q} \begin{pmatrix} | & | \\ h_{\infty,(i+1)} & h_{\infty,(i-1)} \\ | & | \end{pmatrix} m \begin{pmatrix} — & h_{\infty,(i+1)} & — \\ — & h_{\infty,(i-1)} & — \end{pmatrix} + h_{\infty,i}^t h_{\infty,i} \end{aligned} \tag{4.2}$$

where $h_{\infty,i}$ has to be normed such that $h_{\infty,i} p_{c,i} = 1$. Then $q, \bar{q}$ and m can be chosen such that the biquadrics are reproduced correctly. These parameters are the same for every $i \in \{1, 2, 3\}$. The form of equ. 4.2 is identical to that of equ. 2.5 with $l_1 = h_{\infty,(i+1)}$ and $l_2 = h_{\infty,(i-1)}$. Furthermore $h_{\infty,(i+1)} \times h_{\infty,(i-1)} = p_{c,i}$.

Remark 11. For higher prime numbers than 5 the application of this mecha-

nism to find ambiguous representation matrices is not possible anymore, because already for $q = 7$ each quadric consists of 8 points. If now $\{p_{\infty,1}, p_{\infty,2}\}$ are removed from one quadric $\mathcal{Q}_M$, the point set of the biquadric is invariant, but still 6 points remain in $\mathcal{Q}_M$. As motivated above, already 5 points determine a quadric uniquely. This means that there is no possibility of choosing appropriate points $\{p_1, p_2\}$ from the partner quadric $\mathcal{Q}_{\overline{M}}$ such that $\{p_1, p_2\} \cup \mathcal{Q}_M \setminus \{p_{\infty,1}, p_{\infty,2}\}$ define a quadric if $\{p_{\infty,1}, p_{\infty,2}\} \cap \{p_1, p_2\} = \emptyset$, because the remaining 6 points of $\mathcal{Q}_M$ determine $\{p_{\infty,1}, p_{\infty,2}\}$ uniquely.

4.2.2 Lorentz-Transformations on the point set

In previous work ([Rei16]) *Lorentz-Transformations* were introduced as projectivities that leave the representation matrices invariant. Obviously, such a transformation also leaves the point set invariant. But we already know that non-unique biquadrics exist - so the question arises whether a new kind of Lorentz-Transformations has to be considered that changes between the different representation matrices? Before this question is answered a quick overview over what has been done in the field of Lorentz-Transformations so far is important.
The transformations needed to leave the representation matrices invariant can be calculated quickly at least in 2 projective dimensions, depending on whether the biquadric is hyperbolic or elliptic.

- **Elliptic case:** For the elliptic case we only consider fields $\mathbb{F}_q$ where $-1 \in \overline{\Omega}$, such that an elliptic biquadric can always be brought to the simple form

$$\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \bar{q} & 0 & 0 \\ 0 & \bar{q} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right).$$

  Since a biquadric determines a unique center p_c and hyperplane h_∞ the transformations necessarily have the form

$$\hat{\Lambda} = \begin{pmatrix} \mathbf{\Lambda} & \vec{0} \\ \vec{0}^t & 1 \end{pmatrix}.$$

with $\mathbf{\Lambda}^{-t}A\mathbf{\Lambda}^{-1} = A$, where A is the small submatrix of the biquadric. ([Mec17], p.4)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda & \sigma\sqrt[\mathbb{F}_q]{1-\lambda^2} \\ -\sigma\mu\sqrt[\mathbb{F}_q]{1-\lambda^2} & \mu\lambda \end{pmatrix}$$

where $\sigma, \mu \in \{-1, 1\}$ and $\lambda \in \mathbb{F}_q$. Not all of the used square roots exist in $\mathbb{F}_q$, but it can be shown that always $2(q+1)$ such transformations exist.[9]

- **Hyperbolic case** The problem is very similar to the elliptic case, just the standard biquadric from which we start is different. As $1 \in \mathfrak{Q}(\mathbb{F}_q)\forall\mathbb{F}_q$ the biquadric

$$\left(\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -\bar{q} & 0 & 0 \\ 0 & \bar{q} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

is always hyperbolic. A similar calculation with the substitution $a = \frac{1}{2}(\lambda + \frac{1}{\lambda})$ leads to the following transformation matrix:

$$\mathbf{\Lambda}^{-t} = \begin{pmatrix} \frac{1}{2}(\lambda + \frac{1}{\lambda}) & \frac{\sigma}{2}(\lambda - \frac{1}{\lambda}) \\ \frac{\mu\sigma}{2}(\lambda - \frac{1}{\lambda}) & \frac{\mu}{2}(\lambda + \frac{1}{\lambda}) \end{pmatrix}$$

where $\sigma, \mu \in \{-1, 1\}$ and $\lambda \in \mathbb{F}_q \setminus \{0\}$. For the hyperbolic case it can be shown that $2(q-1)$ solutions exist.

Exchange of single or multiple points

An interesting question is whether projectivities exist such that the biquadric as a point set is invariant, but the points of the two quadric partners

[9] The square root $\sqrt[\mathbb{F}_q]{x}$ is taken in the prime field $\mathbb{F}_q$, not in the real numbers. That means, for a given $x \in \mathbb{F}_q$, the square root $\sqrt[\mathbb{F}_q]{x}$ is another element $y \in \mathbb{F}_q$, such that $y \cdot_{\mathbb{F}_q} y = x$.

mix. This would result in a new class of Lorentz-Transformations which will be called *pseudo Lorentz-Transformations* in the following. These would not hold the representation matrix pair invariant, because the point sets of the single quadrics are not invariant. This is in some sense the analogue to the question whether point sets define their representation matrices uniquely, just asked in terms of transformations. As shown in the previous section we found such non-unique biquadrics only for $q \leq 5$, therefore we expect to find pseudo Lorentz-Transformations only for these cases as well, otherwise the theory would be inconsistent.
Indeed for hyperbolic biquadrics in $\mathbb{P}^2\mathbb{F}_5$ one finds 24 pseudo Lorentz-Transformations, what is exactly three times as much as one would expect for the standard Lorentz-Transformations.[10] But as we already investigated this case in the previous section, the additional transformations are easy to explain. What these special projectivities do, is mapping the points at infinity of one quadric onto some suitable points of the second quadric to define a new hyperplane at infinity, just as we have seen it explicitly before in the example.[11] Fig. 4.3 shows the mechanism once again. In the cases considered until now always 2 points of the quadrics get exchanged - but why is not possible to exchange 4, 6, or more points?[12] Indeed the following theorem will show that for higher prime numbers it is not possible to exchange any number of points between the two quadrics, only exchanging the whole quadric is sometimes[13] allowed.

Theorem 4.2.3. *Let $(M, \overline{M})$ be a pair of representation matrices for a biquadric $\mathcal{B}_{(M,\overline{M})} = \mathcal{Q}_M \cup \mathcal{Q}_{\overline{M}}$. Then it is not possible for $\mathbb{P}^2\mathbb{F}_{q>7}$ to find projectivities $\Lambda \in Aut(\mathbb{P}^2\mathbb{F}_q)$, such that $\mathcal{B}_{(M,\overline{M})}$ is invariant under the action of Λ but $\mathcal{Q}_M \neq \mathcal{Q}_{M'}$ and $\mathcal{Q}_{\overline{M}} \neq \mathcal{Q}_{\overline{M'}}$, except the case $M' = \overline{M}$ and $\overline{M}' = M$. Where $\overline{M}' = \Lambda^{-t}\overline{M}\Lambda^{-1}$ and $M' = \Lambda^{-t}M\Lambda^{-1}$.*

[10] For a hyperbolic biquadric in $\mathbb{P}^2\mathbb{F}_5$ 8 standard Lorentz-Transformations should exist.
[11] Unfortunately no mechanism could be found that clarifies how to construct these transformations that change the representation matrix and leave the point set invariant.
[12] The amount has to be an even number, because the with one point always its symmetric partner has to be transformed to keep symmetry.
[13] This will be investigated in the next section.

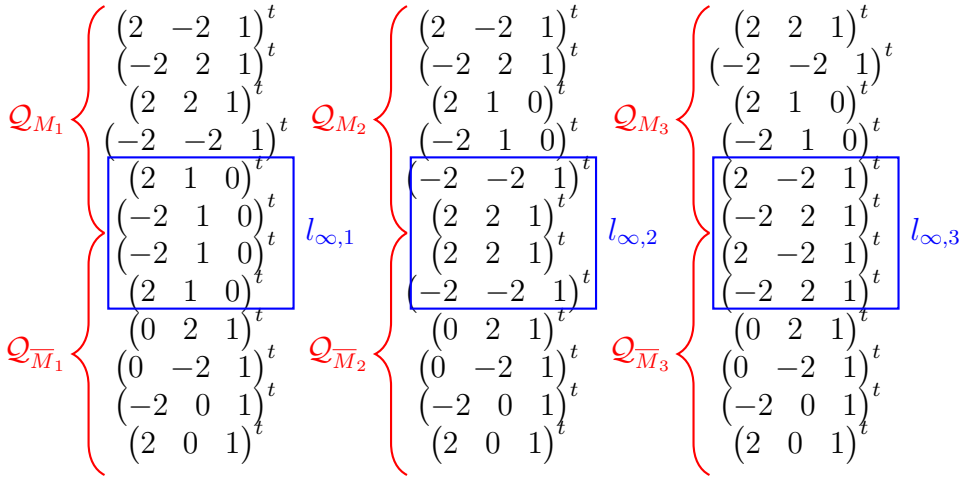


Figure 4.3: For each of the 3 cases 8 standard Lorentz-Transformations exist, hence in total 24. The points in the blue rectangle define the new line at infinity l_∞.

Proof. This theorem will be proven by reductio ad absurdum, therefore we let Λ be a projectivity that holds $\mathcal{B}_{(M,\overline{M})}$ invariant, but $\mathcal{Q}_M \neq \mathcal{Q}_{M'}$ and $\mathcal{Q}_{\overline{M}} \neq \mathcal{Q}_{\overline{M}'}$. Furthermore $M' \neq \overline{M}$ and $\overline{M}' \neq M$.
Suppose $q > 7$. Every biquadric has $2(q+1)$ (elliptic) or $2q$ (hyperbolic) many points. As $\mathcal{Q}_M \neq \mathcal{Q}_{M'}$ and $\mathcal{Q}_{\overline{M}} \neq \mathcal{Q}_{\overline{M}'}$, but the biquadric as a whole is unchanged, there has to exist a certain number of points that is exchanged between the two single quadric partners. Furthermore it is not possible that all points are exchanged, because $M' \neq \overline{M}$ and $\overline{M}' \neq M$. The number of exchanged points always has to be even, because for each point its point-symmetric partner has to be chosen as well to preserve symmetry of each quadric. Now there are two possibilities.

1: If one chooses 6 or more points in each quadric to be exchanged between the quadrics, these points already define all other points of their quadric uniquely. As the point sets of $\mathcal{Q}_M$ and $\mathcal{Q}_{\overline{M}}$ have at most 2 points in common[14], the chosen points from $\mathcal{Q}_M$ are not

[14]Only in the hyperbolic case. In the elliptic case no points are contained in both.

compatible[15] with the remaining points of $\mathcal{Q}_{\overline{M}}$ and vice versa.

2: If one chooses less than 5 points in each quadric to be exchanged, more than 5 points remain in each quadric and define it uniquely. Therefore it is not possible to insert 2 or 4 points from $\mathcal{Q}_M$ into $\mathcal{Q}_{\overline{M}}$ and vice versa, because they are not compatible. For hyperbolic biquadrics it is of course always possible to exchange the 2 points at infinity as both quadrics contain them at the same time - but this case is excluded by the requirements $\mathcal{Q}_M \neq \mathcal{Q}_{M'}$ and $\mathcal{Q}_{\overline{M}} \neq \mathcal{Q}_{\overline{M'}}$.

In conclusion it is not possible to construct two new quadrics that lead to the same biquadric (as a point set) by exchanging points. But that was exactly what we were assuming, therefore this leads to a contradiction and the theorem is proven. □

The case where all points of $\mathcal{Q}_M$ and $\mathcal{Q}_{\overline{M}}$ are exchanged is excluded here, because this case deserves a special treatment due to its interesting properties. The next subchapter will investigate this further.

Remark 12. Theorem 4.2.3 makes clear that for all prime numbers bigger than 7 no transformation exists that exchanges a certain number of points between the quadric partners. This has a strong effect on the level of biquadrics as well - it states that it is not possible to find different representation matrix pairs for the same biquadric[16], because these would necessarily have to be connected via projectivities that do not exist. For $\mathbb{P}^2\mathbb{F}_{q>7}$ the pair of representation matrices of a biquadric is unique.

For prime numbers $q \leq 7$ the theorem cannot be applied, as the quadrics do not contain enough points. But as we are interested in very large prime fields as a foundations of finite space time this is not too critical. But the section about non-unique biquadrics for $q = 5$ also gives strong incidence that it is not possible to exchange more than two points in these cases ($q \leq 7$) as well.

[15] In the sense that they form a quadric together.

[16] This is due to the uniqueneness of representation matrices of quadrics under these conditions.

Exchange of both quadrics as a whole

The case we explicitly excluded above is the exchange of all points at the same time.

$$(\mathcal{Q}_M, \mathcal{Q}_{\overline{M}}) \leftrightarrow (\mathcal{Q}_{\overline{M}}, \mathcal{Q}_M) \tag{4.3}$$

$$\Leftrightarrow (M, \overline{M}) \leftrightarrow (\overline{M}, M) \tag{4.4}$$

Obviously this procedure does not violate one of the uniqueness theorems given during the last chapter, therefore it should be possible to find such transformation matrices. But surprisingly, simulations do not always find this kind of transformation. The existence depends strongly on the used biquadric and the chosen prime field.
To calculate how the transformations would have to look like, we look at elliptic and hyperbolic biquadrics separately once again.

- **Hyperbolic case:** We consider the biquadric

$$\left(\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -\bar{q} & 0 & 0 \\ 0 & \bar{q} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right).$$

  Exchanging all points of the two quadrics corresponds to finding a transformation $T \in \mathrm{Aut}(\mathbb{P}^2\mathbb{F}_q)$, such that $T^{-t}MT^{-1} = \overline{M}$ and $T^{-t}\overline{M}T^{-1} = M$. Again the problem can be reduced to 2×2-matrices.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} -a^2 + b^2 & -ac + bd \\ -ac + bd & -c^2 + d^2 \end{pmatrix} \overset{!}{=} \begin{pmatrix} -\bar{q} & 0 \\ 0 & \bar{q} \end{pmatrix}$$

  So the problem reduces to solving 3 equations for a, b, c, d. Already at this point it is clear that one free parameter has to occur in the final transformation matrix. Solving the three equations one after another gives the result

$$\begin{aligned} a &= \eta\lambda \\ b &= \sigma \sqrt[\mathbb{F}_q]{-\bar{q} + \lambda^2} \\ c &= \frac{\sigma\mu}{\eta} \sqrt[\mathbb{F}_q]{-\bar{q} + \lambda^2} \\ d &= \mu\lambda. \end{aligned}$$

Where $\lambda \in \mathbb{F}_q$ is the free parameter and $\eta, \mu, \sigma \in \{-1, 1\}$. This set of variables only solves $T^{-t} M T^{-1} = \overline{M}$. To ensure that also $T^{-t} \overline{M} T^{-1} = M$ is satisfied the product has to be calculated explicitly.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -\bar{q} & 0 \\ 0 & \bar{q} \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} -\bar{q}^2 & 0 \\ 0 & \bar{q}^2 \end{pmatrix} \overset{!}{=} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

This can only be true if $\bar{q} = 1$ or $\bar{q} = -1$. As $\bar{q}$ has to be a non-square the case $\bar{q} = 1$ can be rejected immediately. The case $\bar{q} = -1$ can only be true in prime fields where -1 is a non-square. Furthermore the square root $\sqrt[\mathbb{F}_q]{q - \lambda^2}$ has to exist.

- **Elliptic case:** The same calculation as above, just with the standard elliptic biquadric pair leads to the transformation matrix

$$T^{-t} = \begin{pmatrix} \eta\lambda & \sigma \sqrt[\mathbb{F}_q]{\bar{q} - \lambda^2} \\ \frac{-\mu\sigma}{\eta} \sqrt[\mathbb{F}_q]{\bar{q} - \lambda^2} & \mu\lambda \end{pmatrix}.$$

  As above this does only fulfill $T^{-t} \overline{M} T^{-1} = M$ if $\bar{q}^2 = 1$ and therefore leads to the same restriction as above.

The calculation shows that only for very special biquadrics a transformation exists that exchanges all the points of the single quadrics with each other. And indeed this is exactly what simulations show.

Example 4.2.2 (Point exchange)**.** In $\mathbb{P}^2\mathbb{F}_7$ the biquadric

$$(M_1, \overline{M}_2) = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

is elliptic. The used non-square $\bar{q}$ is -1, therefore all requirements from above are fulfilled, and several transformation matrices should exist, depending on the value of the free parameter λ and the chosen signs η, μ, σ. $\sqrt[\mathbb{F}_7]{-1 - \lambda^2}$ can take the values

λ	$\sqrt[\mathbb{F}_q]{-1-\lambda^2}$
-3	2 or 5
-2	3 or 4
-1	no solution
0	no solution
1	no solution
2	3 or 4
3	2 or 5

The simulation gives 32 Lorentz-Transformation, 16 of those do not exchange points between the quadrics, but just permute the points inside each quadric in a ordered way.[17] The other 16 Lorentz-Transformations are built from the combinations of λ and $\sqrt[\mathbb{F}_7]{-1-\lambda^2}$ and exchange all points between the quadrics in 16 different ways. Remarkably for all simulations until now there were exactly as many standard Lorentz-Transformations as there were exchange transformations, if they existed. The explanation for this is speculative, but could be that one can choose a 'fundamental' exchange transformation and construct all remaining exchange transformations by the composition of the fundamental one with all standard transformations. This would explain why the number of exchange and standard transformations is equal. Due to time reasons this could not be checked in this thesis, but should be investigated in the future.
As a counterexample, the biquadric

$$\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}3&0&0\\0&3&0\\0&0&1\end{pmatrix}\right)$$

with $\bar{q} = 3$ does not fulfill $\bar{q}^2 = 1$ and therefore only the 16 standard Lorentz-transformations exist.

[17] These are the standard Lorentz-transforamtions investigated in [Rei16].

4.2.3 Classes of Lorentz-Transformations (rotations, point-symmetric, axial-symmetric)

The action of Lorentz-transformations on the point set can be examined by explicitly plotting the biquadric points and the points to which they are mapped by the transformation. One finds three different classes of such mapping: axial symmetric, point-symmetric and rotational ones. The following three cases were not examined in detail, only short examples are given that make the action of Lorentz-transformations visible.For all examples in $\mathbb{P}^2\mathbb{F}_5$ the elliptic biquadric

$$\left(\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

was used.

Point-symmetric transformations

As the points of each quadric are point-symmetric to the center point and do not mix under Lorentz-transforamtions, the only point to which the Lorentz-transforamtions can be point-symmetric is the center point of the biquadric. An explicit example for $\mathbb{P}^2\mathbb{F}_5$ is given in fig. 4.4. The transformation in this example was

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Axial-symmetric transformations

Until now it is unknown for how many and for which axis such transformations exist - this would be a task for further examination in future thesis. One example of axial symmetry is given in fig. 4.5. The transformation shown in fig. 4.5 is given by

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

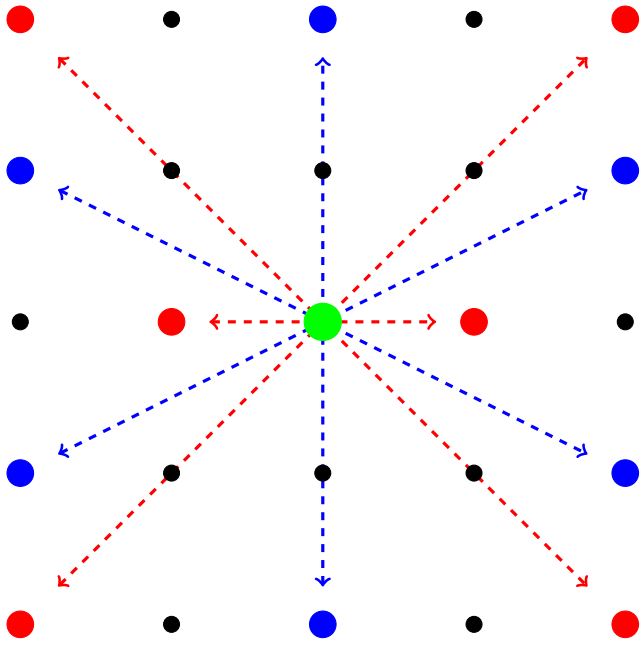


Figure 4.4: Point-symmetric Lorentz transformation

Rotational transformations

Rotation in finite fields is a difficult notion, as no explicit order of the points exists, it is not clear how to define the direction or the angle of a rotation. The word rotation here is inspired by the visible action of rotation matrices as operators on the real vector space. The rotation shown in fig.4.6 is given by the matrix

$$\begin{pmatrix} -2 & -1 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

What makes the word rotation counter-intuitive is that the finite affine plane is periodic, therefore $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$. Plotting the rotation with these points would result in a more 'circle-like' orbit for the red quadric transformation and resemble our intuition of rotation more. This is shown by the additional grey points.

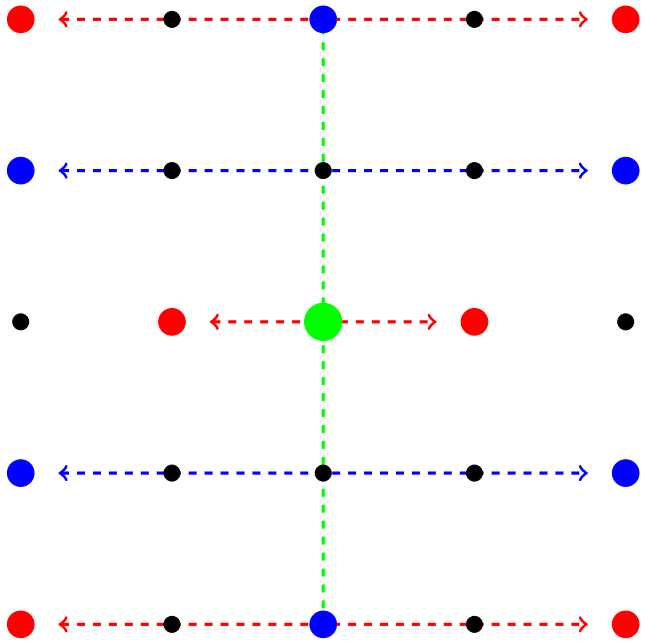


Figure 4.5: Axial-symmetric Lorentz transformation

In contrast to the rotation shown in fig. 4.6 there exist also rotations that can be decomposed into two suborbits of rotations of only 3 (for $q = 5$) points. Furthermore there exist rotations with the contrary rotating sense. But until now it is not known how this relates to the known rotation matrices with sine and cosine in the continuum limit. What makes this even more counter-intuitive is that the rotating sense of the both quadrics seems to be opposing. While one quadric is rotated in positive direction, the other one rotates in negative direction.
This led to the question if there are rotating Lorentz-transformations that have the same rotating sense - and indeed one example was found for an elliptic biquadric in $\mathbb{P}^2\mathbb{F}_7$.
The used biquadric for $q = 7$ was

$$\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

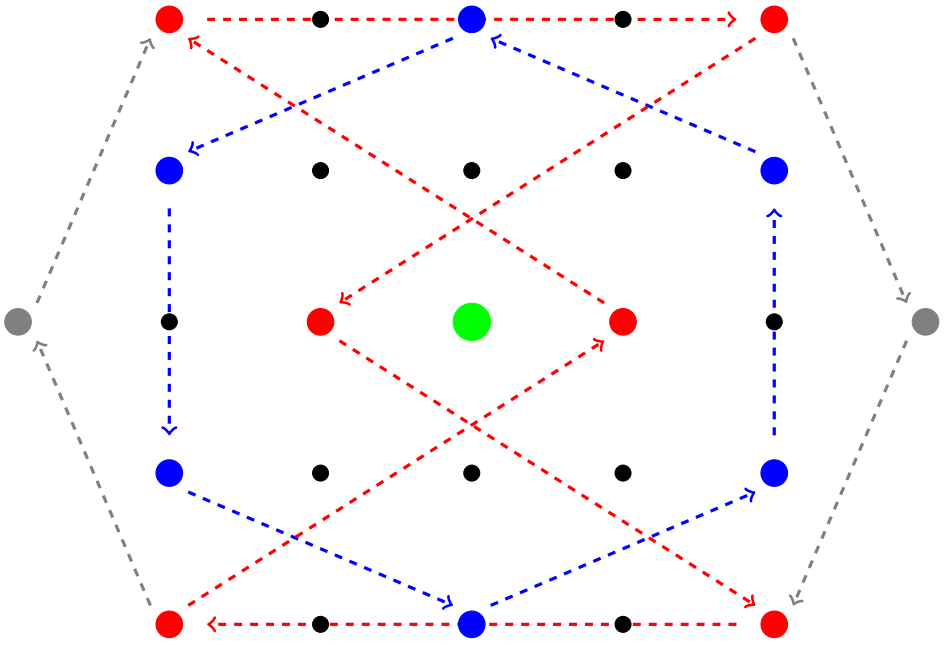


Figure 4.6: Rotational Lorentz transformation with periodically expanded points outside of the projective plane to make rotation clear. These points are shown in grey. Both quadrics rotate in opposing directions.

and the rotation given by

$$\begin{pmatrix} 2 & 2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Until now no further statement can be made on the question what is responsible for the rotating sense of a quadric. Once again it is important to remark that it is absolutely not clear by now what rotational direction means in finite geometries, as the points are not ordered, and therefore it is difficult to give a projectively invariant definition of which point follows which point.[18] The definition of rotating sense in this topic is naively

[18]Of course this can be done naively by defining an order on the point set, but this

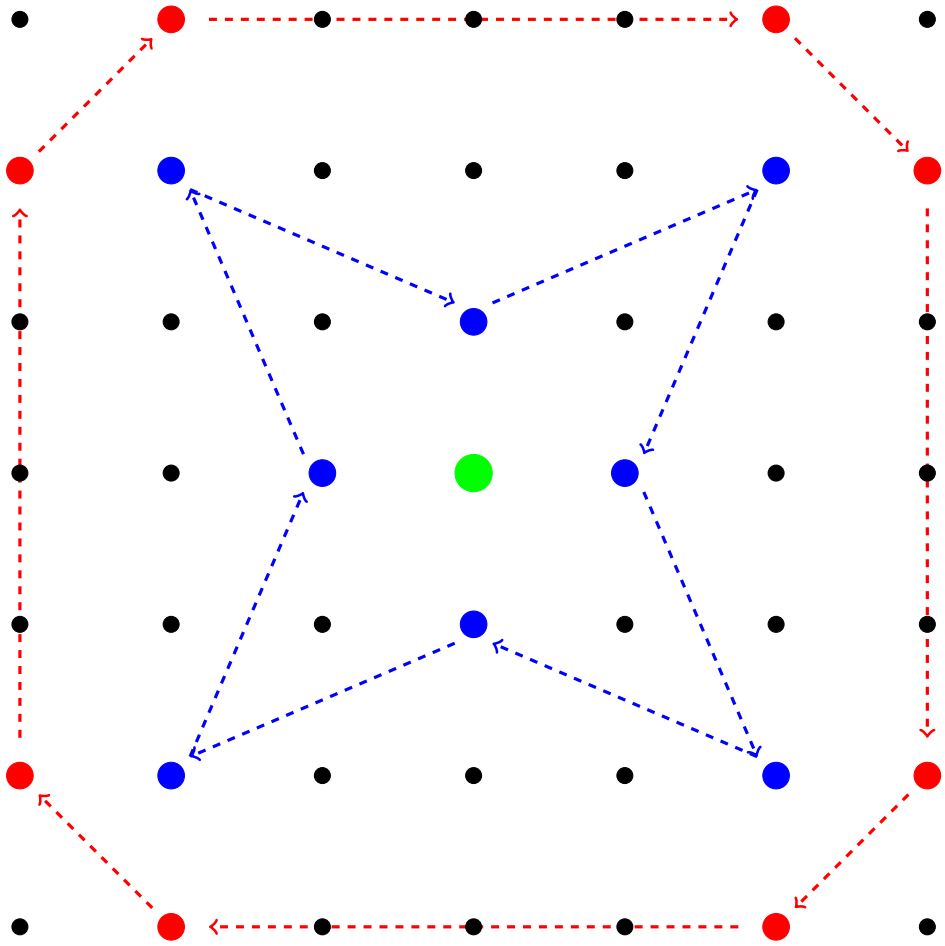


Figure 4.7: Non-opposing rotating sense for an elliptic biquadric in $q = 7$.

deduced from our everyday intuition we get from our continuous, ordered world. Further investigation shall be done on this topic.

order will not be invariant when the points are transformed with projectivities.

5 Cellular automata in finite projective geometries

"Look deep into nature, and then you will understand everything better."

Albert Einstein

The last chapter showed that biquadric fields can sometimes be unintuitive, therefore the idea of this chapter is to study a well-known dynamical system on a new basespace, namely on finite projective geometries instead of a square lattice with a neighborhood relation defined by biquadric fields. By comparing known results with the results from biquadric field simulations, we hope to get some better feeling for biquadric fields. How do they precisely differ from well-known neighborhood structures? What properties do they have? How strongly are they connected with each other? How do they behave if inhomogeneities are introduced?

5.1 Introduction to Cellular Automata

The models we use here are a special type of dynamical systems called *Cellular Automata*. Despite following simple rules they show complex dynamical behavior on long time scales and therefore give the possibility to model complex physical systems and analyze them. Before we give a precise definition, let us consider the history of cellular automata in more detail. It was in the 1940's that *John von Neumann* worked on self-replicating systems in Los Angeles. Following ideas by *Stansislaw Ulam* [Ula52] von Neumann worked in a fully discrete framework made of cells with several binary states, evolving in discrete timesteps. These discrete dynamical systems are nowadays refered to as *Cellular Automata*. Indeed von Neumann

suceeded in constructing a self-replicating dynamical system using these ideas ([Neu66]). Later many more cellular automata were constructed, one of the most famous by the british mathmatician John Conway in 1970, called the *Game of Life.*([Gar70]) This model will be explained in detail later, now we concentrate again on general cellular automata. That it may be possible to simulate the behavior of particles in a gas or fluid using cellular automata was first understood in the 1980s, when it was recognized that the HPP lattice gas models followed indeed the rules of cellular automata. ([CD98], p. 1-5)

Definition 5.1.1. Cellular Automata ([CD98], p. 12) A cellular automata requires

- ◇ a regular lattice of cells covering a portion of a d-dimensional space
- ◇ a tuple $\mathbf{\Phi}(\vec{r},t) = \{\Phi_1(\vec{r},t), ..., \Phi_m(\vec{r},t)\}$ of Boolean variables attached to each site $\vec{r}$ of the lattice and giving the local state of each cell at the time $t = 0, 1, 2, ...(t \in \mathbb{N})$
- ◇ a rule $\mathbf{R} = \{R_1, ..., R_m\}$ which specifies the time evolution of the states $\mathbf{\Phi}(\vec{r},t)$ in the following way

$$\Phi_j(\vec{r},t+1) = R_j(\mathbf{\Phi}(\vec{r},t), \mathbf{\Phi}(\vec{r}+\vec{\delta}_1,t), ..., \mathbf{\Phi}(\vec{r}+\vec{\delta}_q,t))$$

  where $\vec{r}+\vec{\delta}_k$ designate the cells belonging to a given neighborhood of cell $\vec{r}$

Following the definition it is obvious that the system is evolved in discrete time steps, and that the state of a given cell at $\vec{r}$ in step $t+1$ only depends on the chosen rule and on the states of the neighborhood in step t. The chosen rule $\mathbf{R}$ and neighborhood structure is identical for each cell[1] and the total state of each cell is specified by giving the value of each Boolean variable $\Phi_j(\vec{r},t)$ and the evolution rule R_j, which can be different for each

[1] We will investigate non-identical neighborhood structures as well when we investigate inhomogeneous biquadric fields.

of the variables.
In standard 2-dimensional cellular automata on square lattices often two different types of neighborhoods are used ([CD98], p. 14):

1 **Von-Neumann neighborhood:** Only the cells which are directly adjacent in the four geographical directions north, west, east and south are considered as neighbors.

2 **Moore neighborhood:** In addition to the von-Neumann neighborhood it contains also the neighbors to the north-east, north-west, south-east and south-west.

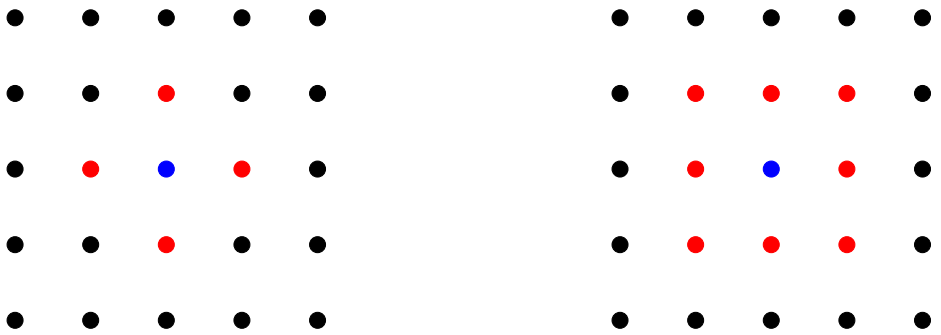


Figure 5.1: Two standard types of 2-dimensional neighborhoods - on the left a von-Neumann-neighborhood, on the right a Moore-neighborhood. Blue indicates the central points, which gets updated by its neighbors shown in red.

It is clear that the chosen neighborhood can be responsible for the behavior of the cellular automate, therefore it is important to check carefully which type of neighborhood is needed before modeling a system. *But as the cellular automate crucially depends on the neighborhood it is also possible to get informations about the properties of the neighborhood by assigning another neighborhood to a well-known cellular automate.* This is what we will do in this chapter. By replacing the Moore neighborhood with a biquadric neighborhood the properties of the cellular automate *Game of Life* change and allow to get information about the biquadric neighborhood.
As one cannot use infinite lattices in practice, it is important to think about the properties of the boundary - 4 different types of boundary conditions

are common: adiabatic, fixed, reflective and periodic boundary conditions. ([CD98], p. 16) During this thesis we only work with periodic boundary conditions, because the finite projective geometry itself is also $(q+1)$-periodic (where q is the chosen prime). The affine plane, in which we are mainly interested during this chapter, is q-periodic. If we leave the affine plane on the top we will enter it again on the bottom, if we leave it on the right or left we will enter again on the other side.
Due to the fixed rule set $\mathbf{R}$ the evolution of such cellular automata is completely determined by the chosen initial configuration and the chosen rules R_j. It is possible, and in some cases useful to introduce stochastic elements into the transition rule. Such cellular automata are called probabilistic, but will not be investigated further in this thesis.

5.2 Example: Cellular Automata on 1-dimensional projective space

To get some intuition how to work with cellular automata a 1-dimensional example shall be analyzed in this section. In the most simple case the cellular automata consists of a certain number N of cells c_i ($i \in \{1, ..., N\}$), all arranged on a line. The transistion rule for each cell c_i from $t_j \to t_{j+1}$ is determined by the states c_{i-1,t_j} and c_{i+1,t_j} of the two neighboring cells. The cells can only have two states: 0 and 1, such that the set of boolean variables attached to each site is given by $\mathbf{\Phi}(\vec{r}, t) = \{\Phi_1(\vec{r}, t)\}$. The rule set $\mathbf{R}$ consists of only one rule R_1, determining $\Phi_1(\vec{r}, t+1)$ depending on the boolean values of the neighboring cells. In total 8 combinations of neighboring states c_{i-1}, c_i, c_{i+1} are possible, for each of these one can choose a transition rule for c_i.[2] That results in 256 cellular automata with binary states on a line, which were classified into different groups of legal and illegal cellular automata by Stephen Wolfram, depending on the complexity and chaotic behaviour they show. [Wol83] Later a more refined classification has been done by grouping the cellular automata in 4 groups

[2] This means assigning either the value 0 or 1 to the middle cell in the next step depending on the neighbors.

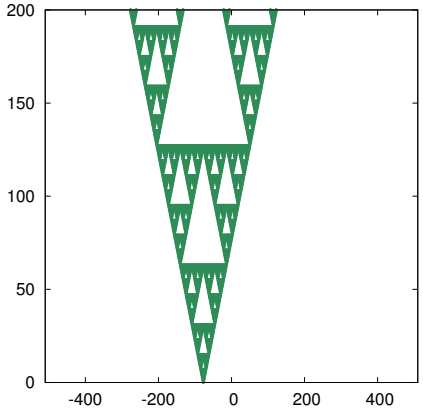
depending on the their long term bahvior. (vgl. [CD98], p. 23)
A typical transition table looks as shown in tab. 5.1. The name of the rule

states at $t = t_j$	111	110	101	100	011	010	001	000
$c_{i,t_{j+1}}$	0	1	1	1	1	1	1	0

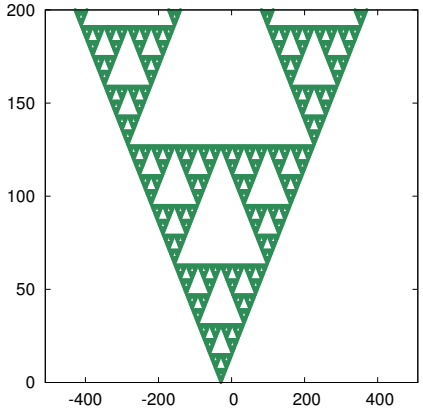
Table 5.1: Example for rule 126, as given by Wolfram's definition of 1-dimensional cellular automata.

is the sequence of transition states expressed in the decimal system. For example 01111110 corresponds to rule 126. Interpreting the line on which the cells live as the affine part of a 1-dimensional finite projective space, it is possible to introduce new neighborhood relations in a ordered way. Assigning a biquadric[3] to each point defines a neighborhood for the specific point. In one dimension each biquadric consists of only 2 points, therefore Wolframs cellular automata can be easily generalized to this projective case.
Usually 1-dimensional cellular automata are analyzed by space-time charts, meaning that the x-axis represents the cells on the line (their position in space), and the y-axis represents the evolution of the initial state in discrete time steps. Usually the initial condition is given by placing one seed at a random position on the line, all other cells are assumed to be in the opposite state. Then the system is evolved according to the transition rule. For rule 126 and periodic boundary conditions the pictures in fig 5.2 occur. Depending on the chosen neighborhood relations the resulting space-time-chart changes, but at least for the case 5.2(b) and 5.2(c) the original evolution pattern can still be seen, even if it gets broader. For random neighborhood relations the original pattern is not visible anymore, the resulting pattern looks more chaotic, but still shows some regularity.

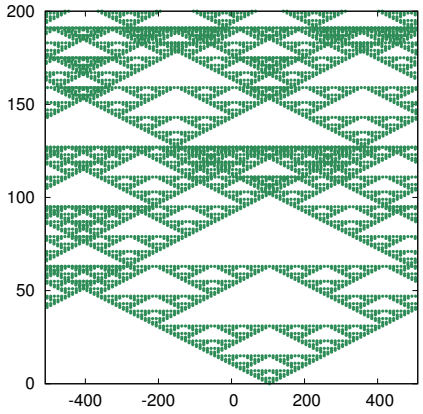
[3]Analogous to higher dimensions, in a 1-dimensional projective space, quadrics are represented by 2×2 matrices, which are normalized to 1 in the lower right entry.



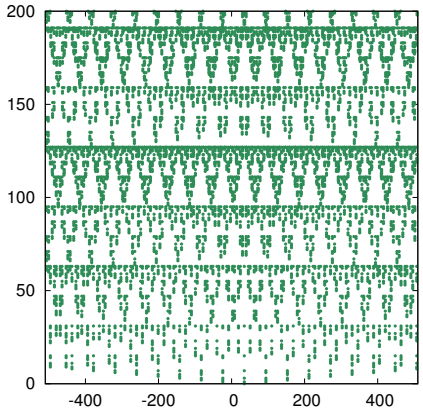
(a) Next point is neighbor.



(b) Two points until the next neighbor.



(c) Ten points until next neighbor.



(d) Randomly chosen neighbors.

Figure 5.2: Development of a 1-dimensional cellular automata following rule 126 under different neighborhood relations.

5.3 Game of Life

Replacing the 1-dimensional structure of the former example by higher dimensional grids results in a much richer dynamical structure, depending on the chosen transition rules. One of the most prominent 2-dimensional cellular automata is Conways *Game of Life*, first developed in 1970 by the British mathematician *John Horton Conway.*([Gar70]) Over the years its

popularity increased continuously, making it one of the most investigated cellular automata so far.

5.3.1 Game of Life with standard neighborhood relations

Before the investigation in projective geometries can start, it is insightful to recover the standard Game of Life in order to be more sensible to the changes due to the used biquadric neighborhood relations.
The Game of Life is based on a 2-dimensional square lattice, with binary cells σ_α. The state $\sigma_\alpha = 1$ is called *alive* and the state $\sigma_\alpha = 0$ is called *dead*. Simple transition rules based on a Moore-neighborhood determine the evolution of the system.

- ◇ A living cell with more than 3 living neighbors will die due to overpopulation.
- ◇ A living cell with only 0 or 1 living neighbors will die due to loneliness.
- ◇ A dead cell will be born in the next step if it has exactly 3 living neighbors.

These rules were chosen carefully to meet three desired properties, as M. Gardner writes in his article for *Scientific American* in 1970 ([Gar70]):

- ◇ There should be no initial pattern for which there is a simple proof that the population can grow without limit.
- ◇ There should be initial patterns that apparently do grow without limit.
- ◇ There should be simple initial patterns that grow and change for a considerable period of time before coming to end in three possible ways: fading away completely (from overcrowding or becoming too sparse), settling into a stable configuration that remains unchanged thereafter, or entering an oscillating phase in which they repeat an endless cycle of two or more periods.

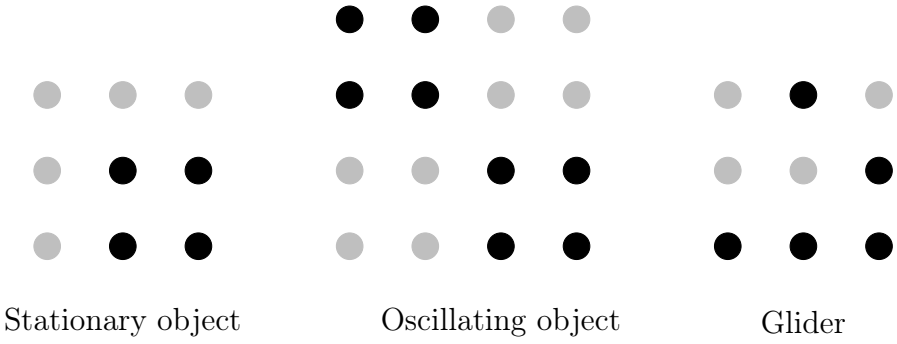


Figure 5.3: Examples for special objects in the standard configuration of Game of Life.

Despite its simple rules the model shows unpredictable behavior. There exist oscillating objects or even objects that move in the underlying square lattice, i.e. gliders. These gliders can also be generated by so-called glider guns. If two gliders hit they can annihilate what makes logical computations possible, the presence of gliders can be interpreted as a logical 1, the absence as a 0. That makes the implementation of calculations possible, even if this is a lot of work. Once started from a random initial configuration the Game of Life evolves over many iterations and will eventually end up in a stationary or oscillating state of density $\rho_{\text{stat}} \approx 0.03$. ([SS78], p. 293-297) The Game of Life is completely deterministic and discrete, as the rules are the same for each cell in each iteration step - therefore it should be possible to predict the stationary state in which it will end just from the random initial configuration. In praxis this is not possible and one has to work with simulation methods.

5.3.2 Game of Life in finite projective geometries

Moving away from the standard Moore-neighborhood and equipping the base-space with biquadric neighborhood relations is the first step towards our generalization of Game of Life and introduces long-range interactions which were not present in the standard case. It is important to remark that all of the following results do only take the affine part $\mathbb{A}^2\mathbb{F}_q$ of the projective space $\mathbb{P}^2\mathbb{F}_q$ into account. Neglecting the line at infinity is a simplification we do here to be as close as possible to the standard game

of life, which is also defined on a quadratic square lattice with periodic boundary conditions.
As each elliptic biquadric consists of $2(q+1)$ points, obviously every point in the affine plane has $2(q+1)$ neighbors. If the field is flat, each point is also contained in $2(q+1)$ neighborhoods. Therefore the number of neighbors depends on the chosen prime number, what makes it necessary to adapt the standard rules of Game of Life to this new situation. The rules are adapted in the following way:

- ◇ If the percentage $f_{\text{alive}} = \frac{N_{\text{alive}}}{2(q+1)}$ of living neighbor points of a living point satisfies $\frac{1}{8} < f_{\text{alive}} \leq \frac{3}{8}$ the point remains alive.

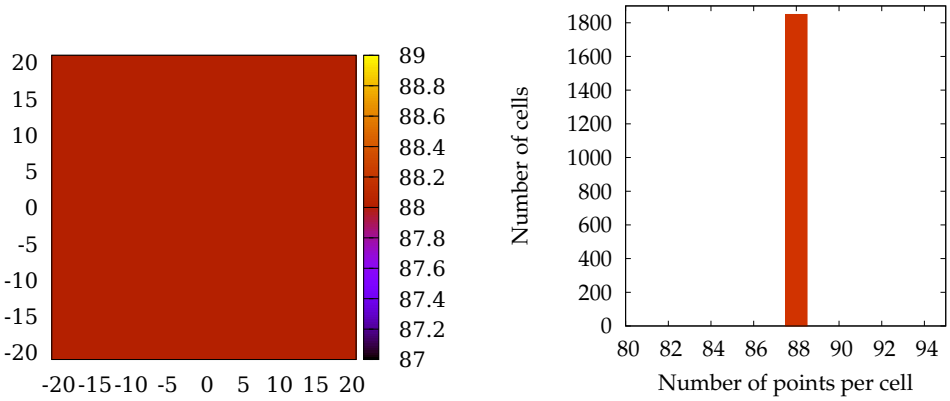
- ◇ If the percentage f_{alive} of living neighbor points of a dead point satisfies $f_{alive} = \frac{3}{8}$ the point is born in the next generation.

- ◇ Living points that do not satisfy one of the first two rules die in the next step.

The similarity to the standard rules is clearly visible, it is just modified to fit the new requirements in a finite projective geometry.
To evolve the Game of Life an algorithm has been written in C++. First for every point in the projective geometry all neighbors are calculated by finding the biquadric points for this site. These are stored in a vector for each point separately, to be easily accessible during the evolution of the cellular automate. To define an initial point configuration two different methods are possible – one can either define a completely random distribution of points, satisfying the requirement that the density of living points in the initial configuration $\rho_{\text{init}} = \frac{N_{\text{init}}}{N_{\text{all}}}$ is equal to some desired manually chosen density. Or one can also define symmetric initial configurations to see how symmetry is affected by the development of the Game of Life - this will get particularly interesting when we study inhomogeneous biquadric fields. Starting from the initial configuration each point is analyzed by counting how many of its neighbors are alive, then its status in the next generation is established by following the given transition rules. When this is done for all points, all points are updated simultaneously to their new state for the next iteration step. This is done until a stationary state is reached.

5.3.3 Homogeneous neighborhood relation

Using homogeneous biquadric fields to define the neighborhood relation is as close as possible to the standard case, because the Moore neighborhood also defines a homogeneous, flat field. It is already known that a flat



(a) Heat map of homogeneous elliptic field (b) Histogram of homogeneous elliptic field

Figure 5.4: Homogeneous flat biquadric field for $q = 43$, shown in terms of heat map and histogram.

biquadric field in the affine space can be generated by translating a given initial biquadric to each point of the affine plane. The emerging field is not only flat, but also fulfills the symmetry condition for biquadric fields. This can be seen easily by

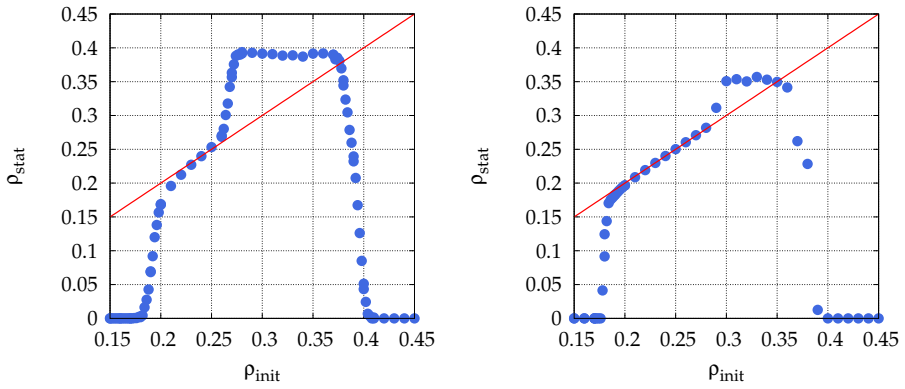
$$\begin{aligned} 0 &\stackrel{!}{=} p^t M_q p \\ q^t M_p q = (T_{q-p}p)^t M_p (T_{q-p}p) &= p^t T_{p-q}^{-t} M_p T_{p-q}^{-1} p = p^t M_q p = 0 \end{aligned}$$

where T_x is a translation matrix and $T_x^{-1} = T_{-x}$. This short calculation makes clear, that if p is a biquadric point of M_q, then q is necessarily a biquadric point of M_p if the two are connected via a translation.

Development of countrate and different regimes

To analyze the long term behavior of the modified Game of Life in terms of density development several simulations were started from a given initial

density and evolved until a stationary state is reached. Then the number of living points in the stationary state was divided by the total amount of points in the affine plane. This procedure was done for a range of initial densities many times, and the stationary final densities were averaged for each initial density. For $q = 31$, this is can be done, but for higher prime numbers the calculation time per step increases rapidly, as not only the number of points that have to be checked increases quadratically, but also the number of neighbors increases linearly with the prime number. For lower prime numbers than $q = 31$ the results had a bad quality, maybe resulting from finite size effects if the lattice is too small. As the number of steps until a stationary state is reached increases strongly with the prime number, it would be handy to be able to predict the density of the stationary state by evolving just one step and looking for fixed points in the development. Taking the average over many such 1-step developments makes the statistics much more convincing and reliable. In the following this is called fixed point method.
The fixed points method has one major disadvantage, it does not take correlations between the points into account that can occur during the evolution because points influence themselves over 2 or 3 steps due to the high connectivity in biquadric fields. In her Master's Thesis, Judith Höfer found that in 2 dimensions all affine points can be reached from every point within at most 3 steps. ([Höf18]) Taking correlations into account will be the main topic of the next subsection - now we will mainly concentrate on density developments that have been simulated by evolving the system until stationarity is reached.
Qualitatively both figures 5.5(a) and 5.5(b) show similar behavior - only in a small range of initial densities from about 0.2 to 0.4 the system evolves to a non-vanishing state. Furthermore inside this range two different regimes occur. Between $\rho_{\text{init}} \approx 0.19$ and $\rho_{\text{init}} \approx 0.27$ the stationary density increases linearly with the initial density, before at about $\rho_{\text{init}} \approx 0.27$ the stationary density jumps to a constant value, which is for $q = 31$ very close to $\rho_{\text{stat}} \approx 0.39$, maybe indicating a phase transition or new state of order at this point. In the simulation for $q = 67$ not all considered configurations were stationary after 2500 iterations, due to the prime number dependent increase in convergence time until stationarity is reached. Nevertheless, the behavior can be deduced quite well from the plot.



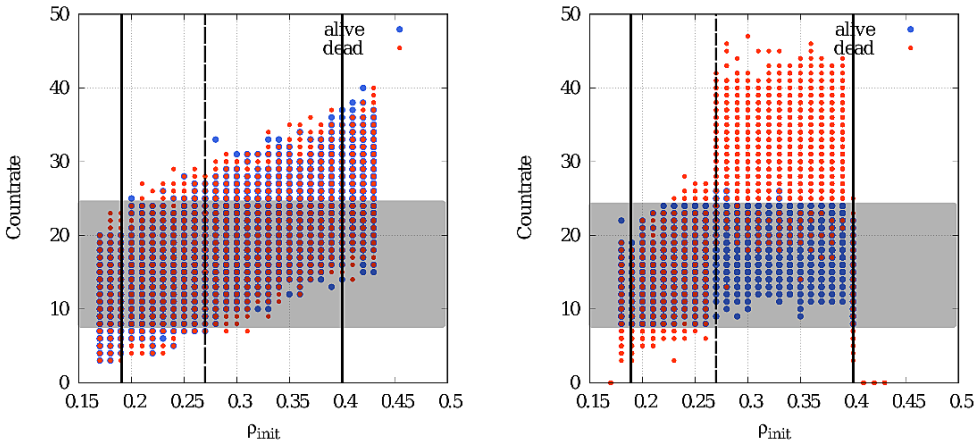
(a) $q = 31$: 500 iterations averaged over 300 *Games of Life*. 4 different regimes are visible: a fixed point regime in the range $0.19 < \rho_{\text{init}} < 0.27$, a stationary regime in the range $0.27 < \rho_{\text{init}} < 0.38$ and two vanishing regimes for all remaining initial densities.

(b) $q = 67$: 2500 iterations averaged over 30 *Games of Life*. Similar regimes occur as for the $q = 31$. The fixed point regime increases and the size of the stationary regime decreases. After 2500 iterations not all considered configurations were stationary. Nevertheless the behavior is clearly visible.

Figure 5.5: Density development for $q = 31$ and $q = 67$ for long runtimes.

The value where exactly the jump happens has to be analyzed further. Therefore, and to analyze the whole regime of non-vanishing density, the countrate[4] for each point is calculated in each step for different initial densities. It is clear that in a stationary state none of the rules given above can be violated, therefore for $q = 31$ no dead point can have exactly 24 living neighbors, because this would lead to a status change in the next iteration step. Additionally, a living point can have neither more than 24 living neighbors, nor can it have less than 8 living neighbors. This is depicted graphically in fig. 5.6 - after 500 iterations nearly all initial states converged. Fig. 5.6 gives also an intuition why there exists only a small range of ρ_{init}, such that $\rho_{\text{stat}} > 0$. This is because for $\rho_{\text{init}} < 0.19$ many initially living points have less than 8 living neighbors and therefore die

[4]Countrate means in the following the number of living neighbors a point has.

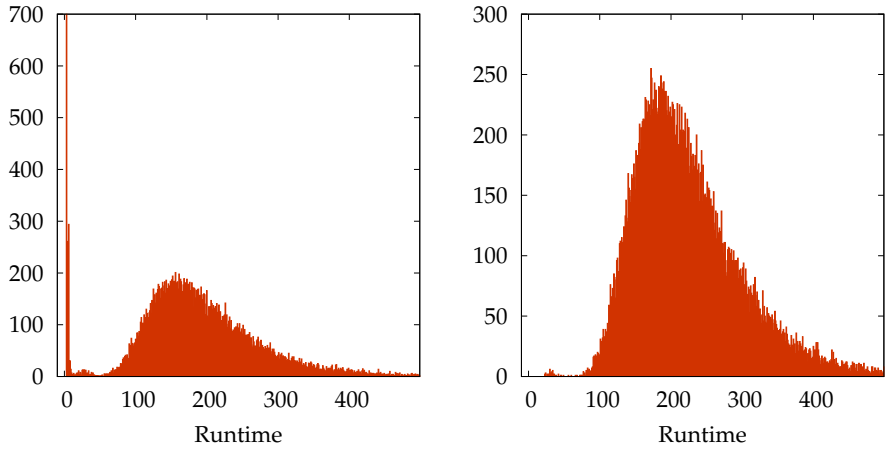


(a) Countrate of initial configuration (b) Countrate after 500 iteration steps

Figure 5.6: Countrate of living points in blue and dead points in red. Living points inside the rectangle do not change their status in the next step, the thick lines indicate the range where the stationary state is non-vanishing, the thin line the jumping point in the density development. After 500 iterations nearly all initial configurations converged, only $\rho_{\text{init}} = 0.27$ and $\rho_{\text{init}} = 0.29$ are still not stationary.

in the next step, effectively decreasing the density of living points even more, such that after only a few steps all cell will die. If $\rho_{\text{init}} > 0.4$ almost all living points have more than 24 living neighbors and will die in the next generation. This will lead to a unpopulated Game of Life within few steps. In the intermediate regime $0.19 < \rho_{\text{init}} < 0.4$ the number of death and birth leads to interesting complex behavior. To explain the jump in the density development plot, one notices that between $0.19 < \rho_{\text{init}} < 0.28$ almost all points lie inside the grey rectangle, therefore these states are already very close to stationarity, what makes them converge fast without a lot of change needed in the configuration. Therefore $\rho_{\text{stat}} \approx \rho_{\text{init}}$. In the range $0.27 < \rho_{\text{init}} < 0.38$ the convergence is much slower, as more points have to change their status to reach stationarity. The stationary density of $\rho_{\text{stat}} \approx 0.39$ cannot be explained by this simple model and has to do with correlations that build between the points during the evolution process.

That the evolution time really depends on the initial density can be seen in a histogram of runtimes T. Therefore two different initial densities were taken – the first one $\rho_{\text{init}} = 0.33$ is inside the stationary plateau regime, while the second one $\rho_{\text{init}} = 0.275$ is located exactly in the jump region. The plots in fig. 5.7 are in accordance with our expectation. $\rho_{\text{init}} = 0.275$ is



(a) $\rho_{\text{init}} = 0.275$: 60000 simulations. (b) $\rho_{\text{init}} = 0.33$: 37500 simulations.

Figure 5.7: Runtime T as a histogram for two different initial densities. For $\rho_{\text{init}} = 0.275$ two different mechanisms contribute: The stationary regime is responsible for the continuum, while the fixed point regime is responsible for the peak contribution.

located exactly inside the jump region, therefore some initial configurations correspond to the fixed point regime range, while others already develop into the stationary regime. Therefore two different contributions can be seen inside the plot - the peak is generated by the very fast converging fixed point regime, while the Poisson-like runtime distribution corresponds to those configurations that converge into the stationary regime. The same Poisson-like distribution occurs also in the histogram for $\rho_{\text{init}} = 0.33$, with a very similar average value of about $\bar{T} \approx 200$, therefore one can conclude that the runtime depends only on the regime in which the corresponding final state lies.

Non-correlated and correlated case

For a theoretical description we follow the construction in [SS78]. The state of cell α at time t is called $\sigma_\alpha(t)$ and the transition rule for each cell is given by

$$\sigma_\alpha(t+1) = \delta\left(0.375 \cdot 2(q+1), \sum{}' \sigma(t)\right) + \sigma_\alpha(t) \sum_{k=0.125\cdot 2(q+1)}^{0.375\cdot 2(q+1)-1} \delta\left(k, \sum{}' \sigma(t)\right) \tag{5.1}$$

where σ is either 0 (dead) or 1 (alive), $\sum'$ is the sum over all neighbors of cell α and $\delta(i,j) = 1$ if $i = j$, otherwise it is 0. The limits of the sum are calculated depending on the prime number and therefore on the number of neighbors.
The density of the whole system is calculated as the mean value of all cell states

$$\rho(t) = \langle \sigma_\alpha(t) \rangle = \frac{1}{M} \sum_\alpha \sigma_\alpha(t)$$

where $\alpha \in \{1, ..., M\}$. Assuming that the density of living squares at time t is $\rho(t)$, the probability to find exactly j living cells in i randomly chosen cells (i is the number of neighbors) is

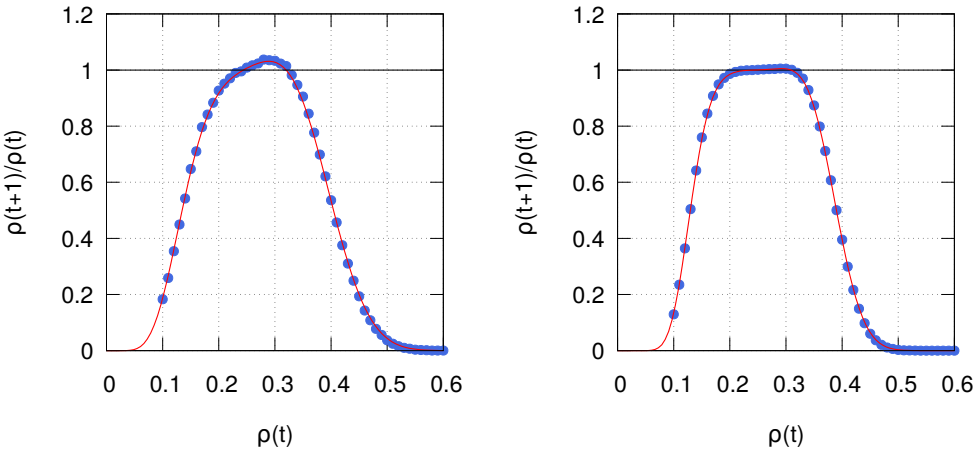
$$\left\langle \delta\left(j, \sum{}' \sigma(t)\right) \right\rangle = \binom{i}{j} \rho^j (1-\rho)^{i-j}.$$

Thus one simply sums up all configurations where the δ is 1 and normalizes with the number of all possible configurations. This can be done for all terms in 5.1 resulting in a theoretical prediction of the density evolution:

$$\begin{aligned} \rho(t+1) = &\binom{2(q+1)}{0.375 \cdot 2(q+1)} \rho^{0.375\cdot 2(q+1)} (1-\rho)^{2(q+1)-0.375\cdot 2(q+1)} + \\ &+ \rho \sum_{k=0.125\cdot 2(q+1)}^{0.375\cdot 2(q+1)-1} \binom{2(q+1)}{k} \rho^k (1-\rho)^{2(q+1)-k} \end{aligned} \tag{5.2}$$

where the sum can be pulled out of the averaging brackets.
By searching for fixed points $\rho(t+1) = \rho(t)$ of this equation it is possible to

calculate the value of the stationary density ρ_{stat} towards which the system will evolve for long times. The accuracy of this result can be checked by simulating just the first evolution step starting from a random initial configuration of Game of Life. The simulated points in fig. 5.8 match very



(a) q=31 (b) q=67

Figure 5.8: Comparison of simulation results with theoretical development of ρ. Uncorrelated theory reproduces uncorrelated development of *Game of Life* very good. Increasing prime number leads to an increase of the fixed point regime (exactly those values where the curve is equal to 1), as already seen earlier.

good with the calculated development of ρ, suggesting that theory and simulations match. The case $q = 31$ shows two different regimes:

- ◇ For $\rho_{\text{init}} < 0.24$ and $\rho_{\text{init}} > 0.32$ the density will vanish in the infinite time limit
- ◇ For $0.24 < \rho_{\text{init}} < 0.32$ the density will increase from step to step until it reaches its fixed point at $\rho_{\text{stat}} \approx 0.32$.

For $q = 67$ the results are very similar, despite the fact that the line is very close to 1 in the regime $0.2 < \rho_{\text{init}} < 0.31$, what would correspond to a nearly continuous regime of fixed points here.
But these theoretical predictions do not match the long term behavior

observed in fig. 5.5(a) which is due to correlations we did not take into account. These correlations between cells occur due to the high connectivity of biquadric fields and are important because points can influence themselves over 2 or more iteration steps, this leads to behavior which deviates from the predicted non-correlated case. Before correlations are treated in detail the question occurs why the simulated data fits the theory almost perfectly, but produces a different long term behavior - for the simulation of this data points only the first iteration step was taken into account. This step starts from a completely random distribution of points, therefore no correlations can exist and the evolution during the first step follows the non-correlated theory. If we look at one step evolutions in a later level of development, the points will not match the predicted curve anymore.
To treat correlations between the cells we follow again the procedure presented by [SS78] starting at page 299. We define a set $L = \{1, ..., l\}$ which contains all neighbors of a given site σ_α and K to be any subset of L with cardinality k. Then the δ-function can be written as a sum over all subsets $K = \{\alpha_1, ..., \alpha_k\}$

$$\delta\left(k, \sum{}' \sigma(t)\right) = \sum_K \sigma_{\alpha_1} ... \sigma_{\alpha_k} \tau_{\alpha_{k+1}} ... \tau_{\alpha_l}$$

where $\tau = 1 - \sigma$. In the appendix of [SS78], starting at page 1, was shown that

$$\left\langle [\sigma_\alpha] \delta(k, \sum{}' \sigma) \right\rangle = \sum_K [\langle \sigma_\alpha \rangle] \langle \sigma_{\alpha_1} \rangle_C ... \langle \tau_{\alpha_l} \rangle_C + \\ + \sum_K \sum_{\text{pairs}} \langle \text{pairs of two} \rangle_C \langle \sigma \rangle_C ... \langle \tau \rangle_C + \text{neglected terms.}$$

The parenthesis around $[\sigma_\alpha]$ indicate that two types of averaged δ-functions exist when $\langle \sigma_\alpha(t+1) \rangle$ is calculated – one with σ_α and one without.
The first sum over K can be evaluated by counting all possible subsets K of cardinality k of L. There are $\binom{l}{k}$ many such subsets, all contribute a summand $\rho^k \mu^{l-k}$, where $\mu = 1 - \rho$ and $\langle \sigma \rangle = \rho$:

$$\sum_K \langle \sigma_{\alpha_1} \rangle_C ... \langle \tau_{\alpha_l} \rangle_C = \binom{l}{k} \rho^k \mu^{l-k}. \tag{5.3}$$

The second summand with the double sum is more complicated, here 3 different pairings are possible for $\langle \text{pairs of two} \rangle_C$.

- $\diamond\ \langle \sigma_{\alpha_i}\sigma_{\alpha_j}\rangle$
- $\diamond\ \langle \sigma_{\alpha_i}\tau_{\alpha_j}\rangle$
- $\diamond\ \langle \tau_{\alpha_i}\tau_{\alpha_j}\rangle$

The contributions of these terms are evaluated in [SS78], starting at page 4 of the appendix for both different types of δ-function averages, resulting in

$$\begin{aligned}\left\langle \delta\left(k, \sum{}' \sigma(t)\right)\right\rangle = \binom{l}{k}\rho^k\mu^{l-k} + \sum_{r,s}\langle\sigma_r\sigma_s\rangle_C \cdot \Bigg[\binom{l-2}{k-2}\rho^{k-2}\mu^{l-k} \\ - 2\binom{l-2}{k-1}\rho^{k-1}\mu^{l-k-1} + \binom{l-2}{k}\rho^k\mu^{l-k-2}\Bigg]\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}\left\langle \sigma_\alpha(t)\delta\left(k, \sum{}' \sigma(t)\right)\right\rangle = \binom{l}{k}\rho^{k+1}\mu^{l-k} + \\ + \sum_{r=1}^{l}\langle\sigma_\alpha\sigma_r\rangle_C\left[\binom{l-1}{k-1}\rho^{k-1}\mu^{l-k} - \binom{l-1}{k}\rho^k\mu^{l-k-1}\right] + \\ + \sum_{r,s}\langle\sigma_r\sigma_s\rangle_C\Bigg[\binom{l-2}{k-2}\rho^{k-2}\mu^{l-k} - \\ - 2\binom{l-2}{k-1}\rho^{k-1}\mu^{l-k-1} + \binom{l-2}{k}\rho^k\mu^{l-k-2}\Bigg].\end{aligned} \tag{5.5}$$

where r and s are both neighbors of the cell we look at.
The important difference between standard neighborhood and biquadric neighborhood is encoded in the number of neighbors and in $\langle\sigma_s\sigma_r\rangle_C$. To evaluate this one uses

$$\langle\sigma_s\sigma_r\rangle_C = \langle\sigma_s(t+1)\sigma_r(t+1)\rangle - \langle\sigma_s(t+1)\rangle\langle\sigma_r(t+1)\rangle \tag{5.6}$$

with $\sigma_r(t+1) = \delta(k, \sum{}' \sigma) + \sigma_r(t)\sum_{\text{allowed } k'}\delta(k', \sum{}' \sigma)$. Thus when computing the product $\langle\sigma_s(t+1)\sigma_r(t+1)\rangle$ one has to compute terms of the form

- ◇ **Term 1**: $\langle \delta^{(s)}(k)\delta^{(r)}(k)\rangle$
- ◇ **Term 2**: $\langle \delta^{(s)}(k)\sigma_r\delta^{(r)}(k')\rangle$
- ◇ **Term 3**: $\langle \sigma_s\delta^{(s)}(k')\sigma_r\delta^{(r)}(k')\rangle$
- ◇ **Term 4**: $\langle \sigma_s\delta^{(s)}(k')\sigma_r\delta^{(r)}(k'')\rangle$

where k is the number of living neighbors needed, such that dead points are born in the next generation and $k', k'' \in [0.125 \cdot 2(q+1), 0.375 \cdot 2(q+1)]$ [5]. The index of the δ-function shows to which cell s or r it refers, and therefore defines the neighbors that one has to take into account to evaluate the δ. The sums over all neighbors of s or r have been omitted as it is implicitly clear that they are there.
Terms of the form 1 and 2 will be evaluated in detail in the following, terms of form 3 and 4 are then easy to evaluate if the principle has been understood properly. One has to distinguish in particular 2 cases:

Case 1: r and s are no neighbors of each other

Suppose s and r have m common neighbors. $\langle \delta^{(s)}(k)\delta^{(r)}(k)\rangle$ is only 1 if r and s have both k living neighbors. It is known that they both have l neighbors in total. Let j be the number of living common neighbors, then there are $\binom{m}{j}$ possibilities to arrange them, for cell s one has $\binom{l-m}{k-j}$ possibilities to arrange the missing $k-j$ living neighbors in $l-m$ neighboring cells. For r the same factor occurs, and one has to sum over all possibilities for j. There are $2k-j$ living cells in total[6] and $2l-2k-m+j$ dead neighbors.[7]

$$\langle \delta^{(s)}(k)\delta^{(r)}(k)\rangle = \sum_{j=0}^{k} \binom{m}{j} \binom{l-m}{k-j}^2 \rho^{2k-j}\mu^{2l-2k-m+j} \tag{5.7}$$

For terms of the form $\langle \delta^{(s)}(k)\sigma_r\delta^{(r)}(k')\rangle$ again j living common neighbors are distributed among m common neighbors. As $k' \neq k$ the summation can

[5] Given by the explicit transition rules
[6] The $-j$ occurs, to avoid double counting of common neighbors for both cells.
[7] The $-m$ also occurs to avoid double counting of common neighbors.

only go to $\min(k, k')$. For cell r there are $\binom{l-m}{k'-j}$ possibilities to distribute $k'-j$ living points among $l-m$ neighbors. For cell s one has $\binom{l-m}{k-j}$. The additional σ_r contributes a factor of ρ, but does not influence the other factors, as s and r are no mutual neighbors.

$$\langle \delta^{(s)}(k)\sigma_r\delta^{(r)}(k')\rangle = \sum_{j=0}^{\min(k,k')} \binom{m}{j}\binom{l-m}{k'-j}\binom{l-m}{k-j}\rho^{k+k'-j+1}\mu^{2l-k-k'-m+j} \tag{5.8}$$

All other terms are evaluated in the same fashion. The additional σ only correspond to factors of ρ. One gets:

◇ 1. term

$$\langle \delta^{(s)}(k)\delta^{(r)}(k)\rangle = \sum_{j=0}^{k} \binom{m}{j}\binom{l-m}{k-j}^2\rho^{2k-j}\mu^{2l-2k-m+j} \tag{5.9}$$

◇ 2. term

$$\langle \delta^{(s)}(k)\sigma_r\delta^{(r)}(k')\rangle = \sum_{j=0}^{\min(k,k')} \binom{m}{j}\binom{l-m}{k'-j}\binom{l-m}{k-j}\rho^{k+k'-j+1}\mu^{2l-k-k'-m+j} \tag{5.10}$$

◇ 3. term

$$\langle \sigma_s\delta^{(s)}(k')\sigma_r\delta^{(r)}(k')\rangle = \sum_{j=0}^{k'} \binom{m}{j}\binom{l-m}{k-j}^2\rho^{2k-j+2}\mu^{2l-2k-m+j} \tag{5.11}$$

◇ 4. term

$$\begin{aligned}\langle \sigma_s\delta^{(s)}(k'')\sigma_r\delta^{(r)}(k')\rangle = \sum_{j=0}^{\min(k'',k')} &\binom{m}{j}\binom{l-m-1}{k''-j-1} \\ &\binom{l-m-1}{k'-j-1}\rho^{k''+k'-j}\mu^{2l-k''-k'-m+j}\end{aligned} \tag{5.12}$$

Case 2: r and s are mutual neighbors

In the previous evaluation we used that s and r are no mutual neighbors for the cases 2,3 and 4. Therefore these terms will change, if s and r are mutual neighbors. For term 1 the same result occurs as in equ. 5.7.
For term 2 the additional σ_r is not just a factor ρ. As cell r is a neighbor of s, $\delta^{(s)}(k)$ only needs $k-1$ more living neighbors to be 1. Again there are $\binom{m}{j}$ possibilities for the common neighbors, j is now in the range $[0, \min(k', k-1)]$. As s already has one living neighbor, there are only $\binom{l-m-1}{k-j-1}$ possibilities for the other living neighbors. For cell r the factor is $\binom{l-m}{k'-j}$. The σ_r is absorbed in the $2l$ neighbors both cells have in total, therefore no additional factor of ρ occurs.

$$\langle \delta^{(s)}(k)\sigma_r\delta^{(r)}(k')\rangle = \sum_{j=0}^{\min(k',k-1)} \binom{m}{j}\binom{l-m}{k'-j}\binom{l-m-1}{k-j-1}\rho^{k+k'-j}\mu^{2l-k-k'-m+j} \tag{5.13}$$

Terms of the form 3 and 4 are evaluated equivalently, counting all possibilities to place j living neighbors on m common neighbors, and the remaining $k^{(\prime)} - j(-1)$ on $l-m(-1)$ neighbors, always taking into account which cells are already alive, due to the σ in the terms. One gets:

- ◇ 1. term:

$$\langle \delta^{(s)}(k)\delta^{(r)}(k)\rangle = \sum_{j=0}^{k} \binom{m}{j}\binom{l-m}{k-j}^2\rho^{2k-j}\mu^{2l-2k-m+j} \tag{5.14}$$

- ◇ 2. term:

$$\langle \delta^{(s)}(k)\sigma_r\delta^{(r)}(k')\rangle = \sum_{j=0}^{\min(k',k-1} \binom{m}{j}\binom{l-m}{k'-j}\binom{l-m-1}{k-j-1}\rho^{k+k'-j}\mu^{2l-k-k'-m+j} \tag{5.15}$$

- ◇ 3. term:

$$\langle \sigma_s\delta^{(s)}(k')\sigma_r\delta^{(r)}(k')\rangle = \sum_{j=0}^{k'-1} \binom{m}{j}\binom{l-m-1}{k-j-1}^2\rho^{2k-j}\mu^{2l-2k-m+j} \tag{5.16}$$

◇ 4. term:

$$\langle \sigma_s \delta^{(s)}(k'') \sigma_r \delta^{(r)}(k') \rangle = \sum_{j=0}^{\min(k'',k')-1} \binom{m}{j} \binom{l-m-1}{k''-j-1} \binom{l-m-1}{k'-j-1} \rho^{k''+k'-j} \mu^{2l-k''-k'-m+j} \tag{5.17}$$

The terms occurring in the evaluation of

$$\sum_{r=1}^{l} \langle \sigma_\alpha \sigma_r \rangle_C$$

where σ_α is the cell we look at, all correspond to this case 2, because σ_α and σ_r are always mutual neighbors[8] by definition.
If the averaged δ's have been computed and summed up, equ. 5.6 is applied to get explicit results for the cumulants, wherefore the uncorrelated expressions for $\langle \sigma_s(t+1) \rangle \langle \sigma_r(t+1) \rangle$ are used. This is inserted into equ. 5.4 and 5.5.
To compute now the cumulants numerically one applies the following steps:

1) Compute all neighbors of a given point

2) Choose one neighborpoint r and iterate over all remaining neighbors s. For each combination do the following steps:

   2a) Check how many common neighbors exist (determine m)

   2b) Check if the two points are neighbors

   2c) Apply suitable combinatorical formulae from above and apply equ. 5.6

3) Do this for all neighbors, neglecting all already known combinations

4) Sum all results

[8] The biquadric field obeys the symmetry condition.

The sum $\sum_{r=0}^{l}\langle\sigma_\alpha\sigma_r\rangle$ is calculated similarly. As already this expansion results in an enormous amount of terms[9] all higher order cumulant contributions were neglected. In fig. 5.9 it is obvious, that the fixed point

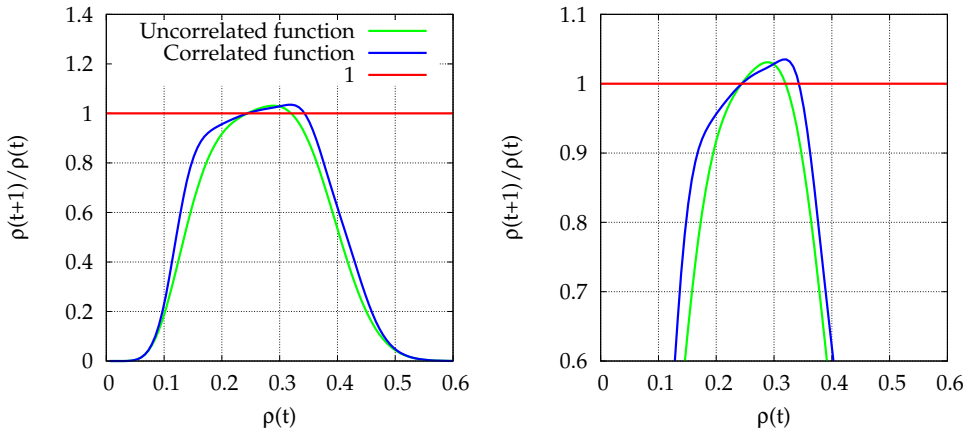
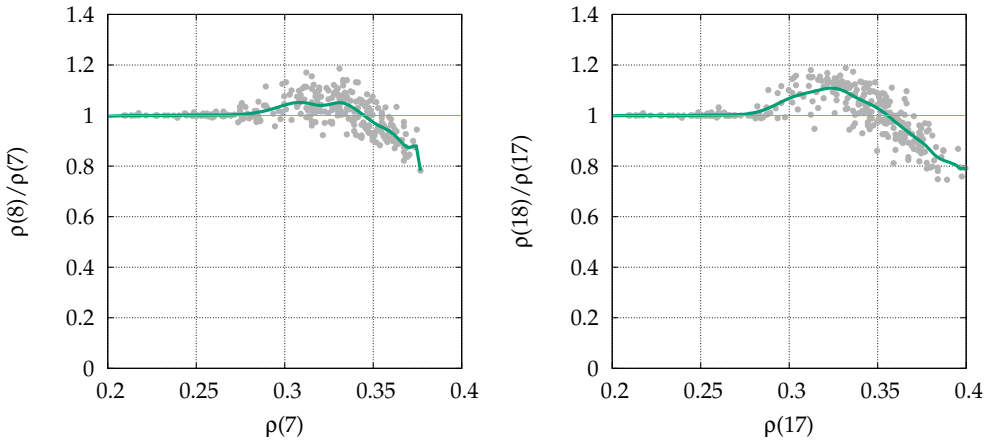


Figure 5.9: Comparison of theoretical prediction with and without second order cumulants. Cumulant expansion up to second order shifts the uncorrelated curve and reproduces simulation results better, but still with some error.

is shifted to higher values of about $\rho_{\text{stat}} \approx 0.35$ what fits the simulated results better, even better results may be possible if higher order cumulants are considered. In the interesting range of $0.2 < \rho_{\text{init}} < 0.3$ the curve gets closer to 1 – if this behavior continues for higher order cumulant expansions and the curve really goes to 1 here, then this would correspond to the continuous fixed point regime observed in the density development plot. As the theoretical computation of higher order cumulants was not further developed in this thesis, an alternative approach has been tried. As before only 1-step evolutions were considered, but not starting at the initial state, but at later steps, e.g. the density development from step 100 to step 101. At these later steps already correlations between the points should have been built, such that the 1-step development has to obey this

[9] Textfile of about 160MB, counting the terms was impossible as the program crashed.

correlated case now. To get good statistics many such simulations have been done and the average smooth Bezier curve was calculated. The result is shown in fig. 5.10. Obviously the fixed points of the density curve shift towards the simulated value $\rho_{\text{stat}} \approx 0.39$ in later iteration steps. This confirms the assumption of correlations between the cells that can not be seen in the random initial configurations.



(a) $7 \to 8$ (b) $17 \to 18$

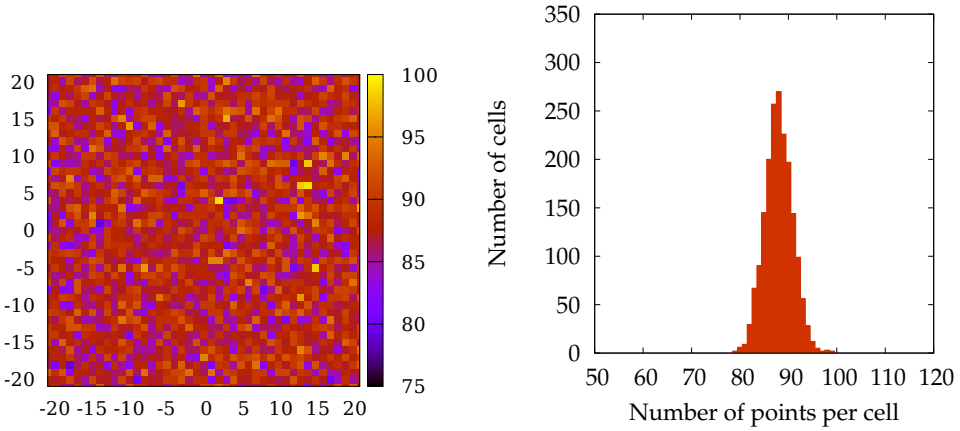
(c) $47 \to 48$ (d) $97 \to 98$

Figure 5.10: Shift of fixed point in the density development due to correlations that built while the Game of Life evolves further and further.

5.3.4 Inhomogeneous neighborhood relation

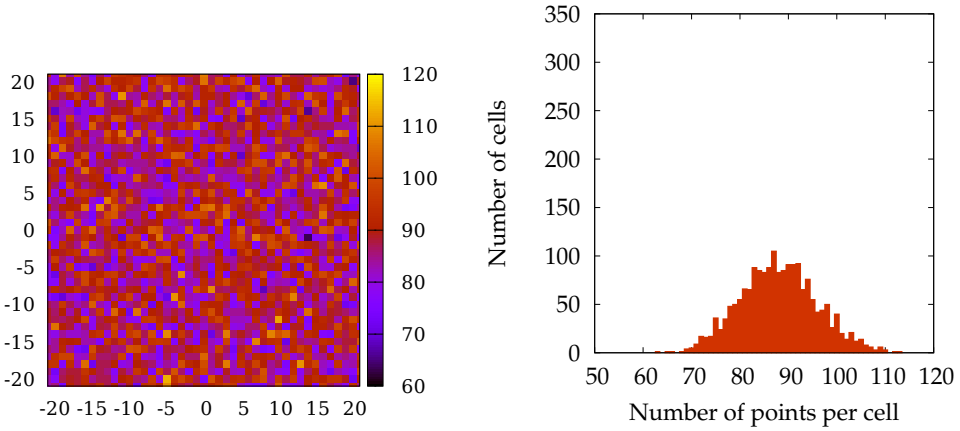
After the analyses of homogeneous biquadric fields, it is interesting to see how their behavior differs from the behavior of inhomogeneous fields. Therefore now inhomogeneous fields[10] are analyzed concerning the *Game of Life*. There are several important differences between homogeneous and inhomogeneous neighborhood relations in terms of biquadrics. While in our homogeneous biquadric fields every point has the same number of neighbors and is neighbor of exactly its own neighbors (symmetry condition fulfilled), inhomogeneous neighborhood relations are much more complex. Still every point has $2(q+1)$ (elliptic case) neighbors in the affine plane, but the neighborhood relations are not symmetric. That means that a point can be neighbor of an arbitrary number of points, no matter if these points are neighbors of the point itself. To analyze the influence of increasing inhomogeneity it is useful to have a inhomogeneity tuning parameter which is implemented in the simulation by first equipping the whole affine plane with a flat translation biquadric field[11]. Then single points are chosen and the biquadric referring to the flat field is removed and an arbitrary biquadric of the same type (elliptic or hyperbolic) is placed there. This tuning is done in steps of single percentages - if one wants to construct an inhomogeneous field, with for example an inhomogeneity I of 0.05 the algorithm chooses $0.05 \cdot q^2$ points and places an arbitrary biquadric there. This results in inhomogeneous fields on which the Game of Life can be implemented in the same fashion as on homogeneous fields. Heat maps and histograms show the influence of different inhomogeneity percentages on the neighborhood structure.
Compared to fig. 5.4 the peak disappears and is replaced by a Gauss-like countrate distribution. Increasing inhomogeneity leads to a broadening of the peak in the histograms. As the total countrate has to be the same in each histogram, the broadening is only possible, if at the same time the height of the peak decreases. In the heat maps this corresponds to a greater range in which the possible values of the count rate are distributed.

[10] Which could be a possible model for a curved spacetime.
[11] As it was done in the former case for flat fields.



(a) Heat map for $I = 0.05$

(b) Histogram for $I = 0.05$



(c) Heat map for $I = 0.5$

(d) Histogram for $I = 0.5$

Figure 5.11: Influence of increasing inhomogeneity on the neighborhood relation. The counter counts the number of neighborhoods in which the point is present. Simulations for $q = 43$ and elliptic fields.

For $I = 0.05$ the highest countrate is 100, whereas for $I = 0.5$ it is 120, for the lowest value of the countrate the equivalent relation is true. In conclusion, increasing inhomogeneity leads to a bigger variation in the number of neighborhoods in which each point can be contained.

Development of symmetric initial configurations

Before we study the density development of inhomogeneous fields, it is insightful to start with a non-random initial configuration, which is highly symmetric. On these configurations the influence of inhomogeneities on the development can be seen directly. For different inhomogeneities I the

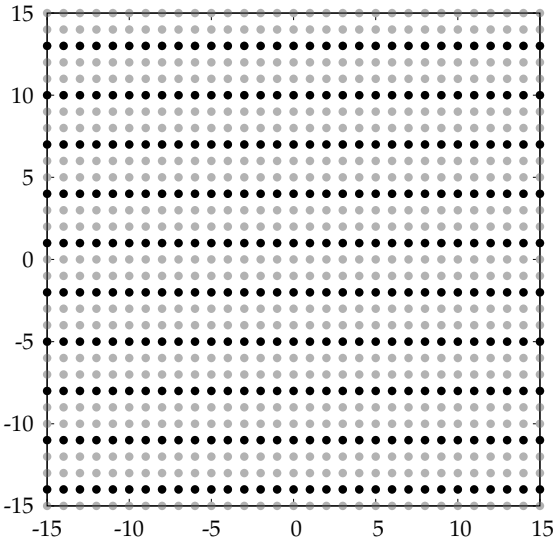
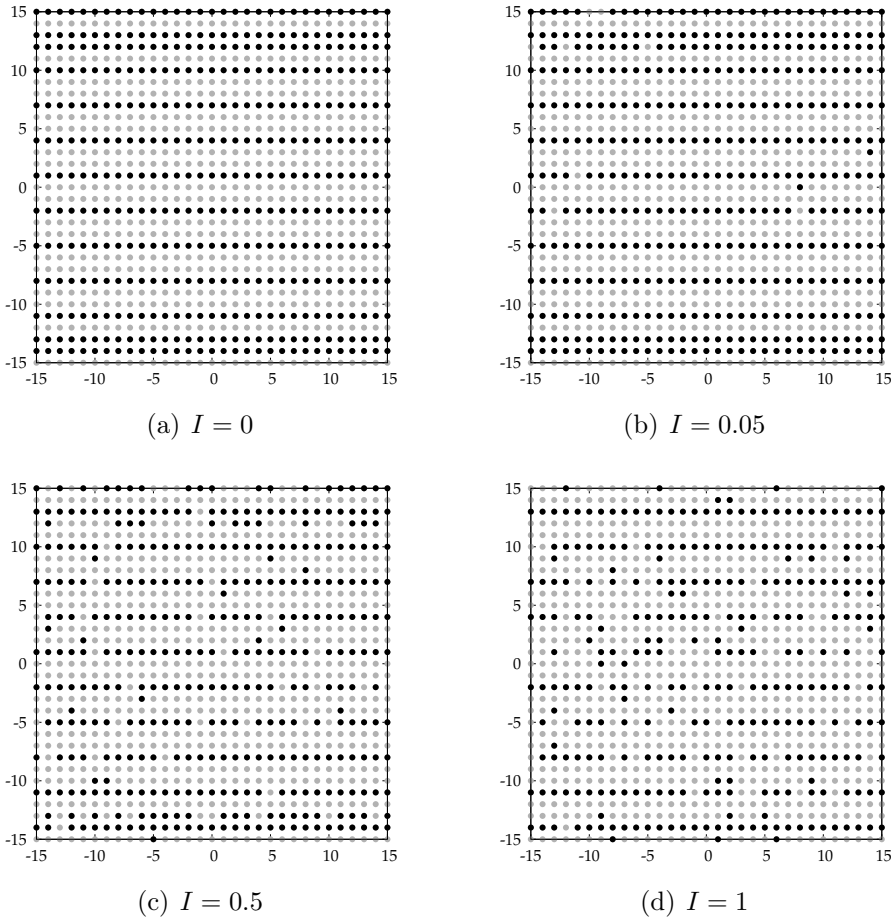


Figure 5.12: Symmetric initial condition.

initial configuration was always chosen equally: one horizontal line is completely filled, followed by two empty lines. This pattern repeats until the whole affine plane is filled with horizontal lines as in fig. 5.12. The Game of Life on a flat biquadric field shows a very symmetric behavior - only whole lines disappear, and only whole lines can appear, as fig. 5.13(a) shows. This can be explained by the construction of the underlying biquadric field: Translating a biquadric translates all its points, therefore if one point on a line has N living neighbors, then every point on this line has N living neighbors, because the chosen initial configuration is translational invariant. For fields with inhomogeneity the initial configuration is still translational invariant, but the biquadric field obeys no translational symmetry. The inserted random biquadrics introduce defects, such that some points are contained in more or less neighborhoods than others. The



(a) $I = 0$ (b) $I = 0.05$

(c) $I = 0.5$ (d) $I = 1$

Figure 5.13: Configurations after developing the initial condition only 1 step further.

behavior of these points differs from the behavior of other points what destroys the symmetric development of the Game of Life. The more inhomogeneity is introduced, the more asymmetric the development is, as already the 1-step developments in fig. 5.13(b), 5.13(c) and 5.13(d) show. Furthermore by analyzing videos one notices that inhomogeneous fields take longer to reach their stationary state - depending on the chosen inhomogeneity.

Countrate development and runtime for inhomogeneous fields

As already done in the last chapter for homogeneous fields, it is also interesting to study the development of stationary densities for inhomogeneous fields. Therefore the same algorithm as before is used, just with an additional inhomogeneity parameter. Obviously the observed behavior differs

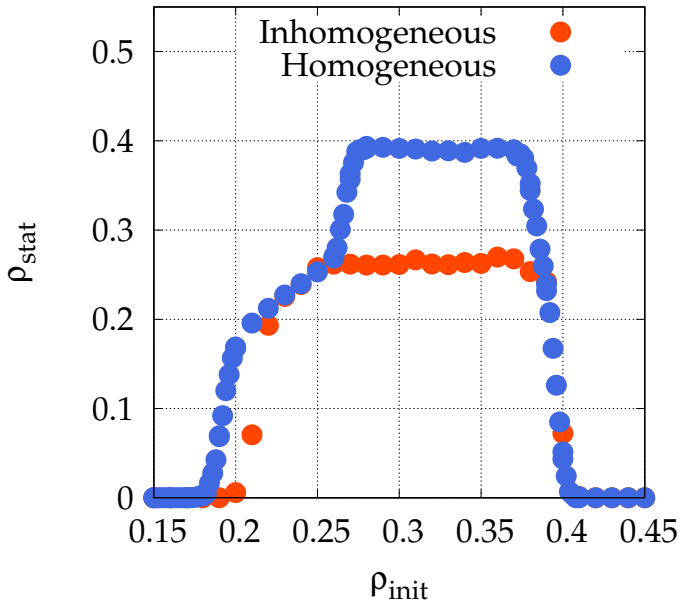


Figure 5.14: Comparison of homogeneous and inhomogeneous density evolution. Prime number was $q = 31$, inhomogeneous points are averages over 250 simulations. 2000 iterations were done for each point and simulation. Inhomogeneity was chosen to be 80%.

in both cases. While in the homogeneous case at $\rho_{\text{init}} \approx 0.27$ a sharp jump can be observed, this jump is not present for inhomogeneous fields. Here also a plateau phase occurs, but at a much smaller ρ_{stat}.
Depending on the introduced amount of inhomogeneity, the runtime until stationarity is reached varies strongly. Plotting runtime and ρ_{stat} over the inhomogeneity I results in fig. 5.15, which shows interesting behavior.

Obviously at $I \approx 0.07$ a fundamental change in the structure of station-

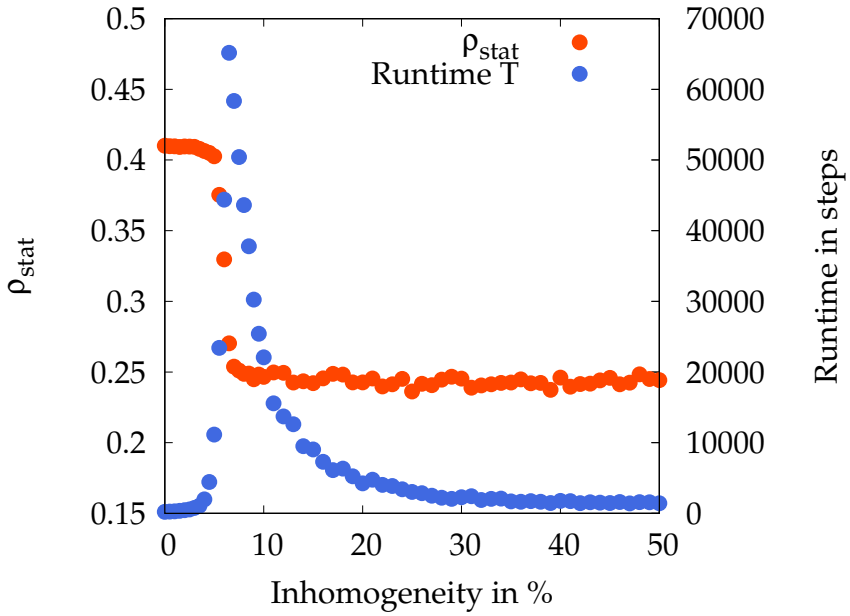
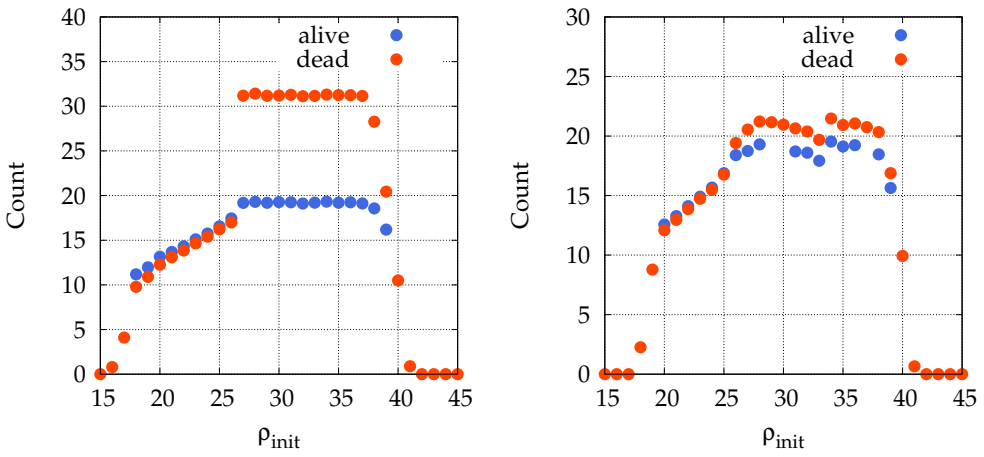


Figure 5.15: Runtime and stationary density as a function of inhomogeneity for $\rho_{\text{init}} = 0.32$. Runtime T shows a peak at $I \approx 0.07$, while stationary density ρ_{stat} jumps at his point to lower values. Maybe this indicates some kind of phase transition due to defects in the system.

ary states takes places. At this point the stationary density ρ_{stat} abruptly jumps from $\rho_{\text{stat}} \approx 0.4$ to $\rho_{\text{stat}} \approx 0.25$, at the same point the runtime as a function of inhomogeneity peaks. For the limiting case of infinite lattice size it seems like the runtime diverges at this point, while the density shows the behavior of a step function, thus indicating a phase transition at this point. It seems like the underlying homogeneous fields can compensate inhomogeneities, if their number does not exceed a critical value. If this critical threshold is crossed. the inhomogeneities get dominant and change the behavior of the Game of Life in the long time limit. Similar behavior can be seen in simpler models, like the Ising model - the Harris-criterion determines if inhomogeneities can influence the critical behavior of statistical models. (vgl. [Har74])

Quite interesting is also to analyze how the introduction of inhomogeneity influences the evolution of the countrate of dead and living points separately. Fig. 5.16 shows that for homogeneous fields the average countrate

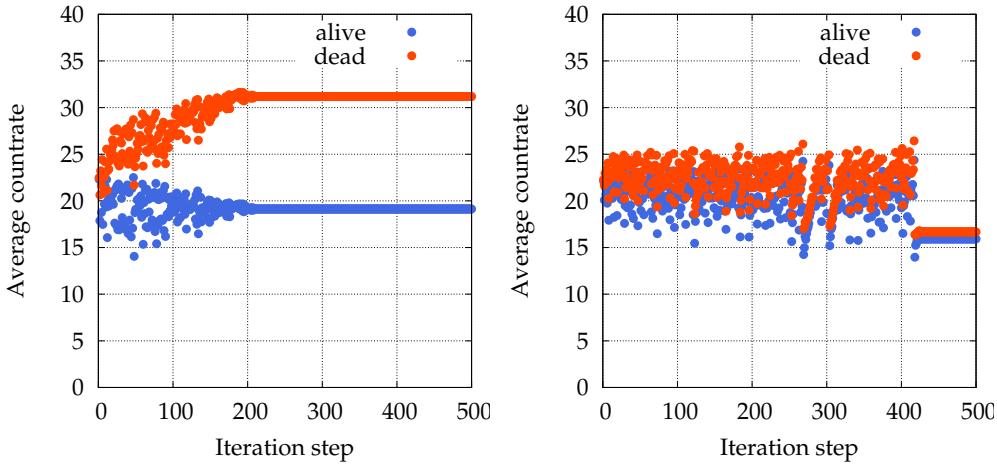


(a) Homogeneous field $I = 0$ (b) Inhomogeneous field $I = 0.5$

Figure 5.16: Comparison of the average countrate of living and dead points for homogeneous and inhomogeneous fields in the stationary state. Averages over 10 Game of life with 500 iterations each. In homogeneous fields the countrate of dead points in the stationary plateau is much higher than that of living points. Introducing inhomogeneity destroys this behavior, now both countrates are nearly equal at a lower level. $q = 31$.

N_{dead} of dead points is higher than for living points. This is in accordance with fig. 5.6, just averaged over all cells. After the introduction of a sufficient amount of inhomogeneity, both countrates N_{dead} and N_{alive} are much closer together at a lower level. This leads to the conclusion that mainly dead points and their neighbors are relevant for the long time evolution of the Game of life, as they have more freedom in the possible number

of living neighbors.[12] Somehow it seems that the introduction of inhomogeneity reduces the number of possible living neighbors each cell can have - maybe because each cell influences a different number of points. It is also insightful to analyze the evolution of a single Game of Life in terms of living neighbors. Again dead and living cells are considered separately. The difference between both plots in fig. 5.17 is obvious, while in the ho-



(a) $I = 0$ (b) $I = 0.5$

Figure 5.17: Average countrate of living and dead cells as a function of iteration steps, starting from $\rho_{\text{init}} = 0.35$. Inhomogeneous and homogeneous fields show a different convergence behavior. While the homogeneous field shows a directed convergence behavior, the inhomogeneous convergence behavior seems rather random.

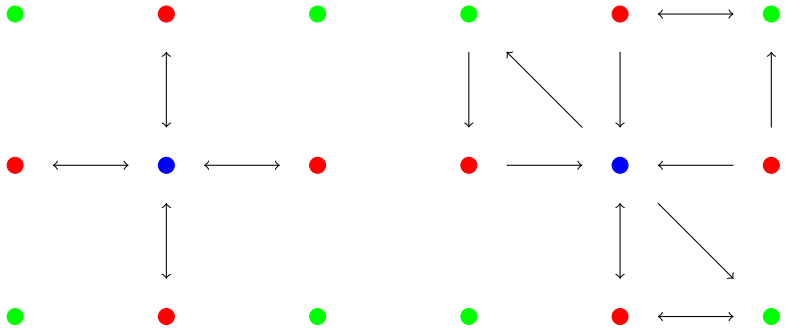
mogeneous case a directed movement towards the stationary countrate can be observed for dead and living points, in the inhomogeneous case more or less random fluctuations are observed, until at some point both countrates get stationary. But in analogy to the homogeneous case, the average coun-

[12] 'Possible' means here all number of neighbors such that the cell does not change its state in the next step.

trate of dead points in the stationary state is always bigger than that of living points.

5.3.5 Properties of the underlying field

In the last section of this chapter the properties of inhomogeneous fields shall be analyzed in more detail. Therefore the Game of Life is not taken into account here, this chapter concentrates merely on properties of the field, i.e. neighborhood relations and symmetry.
Nevertheless one should see why this is important to understand the Game of Life properly. The transition rules in Game of Life depend crucially on the neighborhood relations, therefore fig. 5.18 shows how these can differ due to inhomogeneity. Fig. 5.18 shows an example of neighborhood relations in homogeneous and inhomogeneous fields. While in homogeneous fields everything is symmetric and only first neighbors interact with a given point (due to symmetry) in the inhomogeneous case the neighborhood relations can be much more difficult and complex, such that it is very hard to predict the behavior of the Game of Life. Nevertheless one can study how much the neighborhood relations of inhomogeneous fields differ from the homogeneous one by increasing and decreasing the introduced inhomogeneity. Like that one can tune from a completely flat field to a completely random field.
A first interesting quantity is the percentage of neighboring points that fulfill the symmetry condition, therefore for every point k the neighbors are calculated. For each of the neighbors n one calculates again the neighborhood and searches for the initial point k. If it is contained in the second neighborhood the pair (k, n) is symmetric. The average over the amount of symmetric pairs for each initial point is shown in fig. 5.19(a). As expected for homogeneous fields ($I = 0$), all points are symmetric. But even for completely random fields ($I = 1$) still a small percentage of points fulfills the symmetry relation as the blue curve in fig. 5.19(a) shows. The red curve in Fig. 5.19(a) represents the number of second neighbors a given point k has, that have k as a biquadric point, and therefore are influenced by it. This also means, that these points influence k over two steps, because they are neighbors of the neighbors of k and k influences itself over two steps. For homogeneous fields the plot shows that second neighbors are never influenced by their common center point, therefore the 2-step mutual

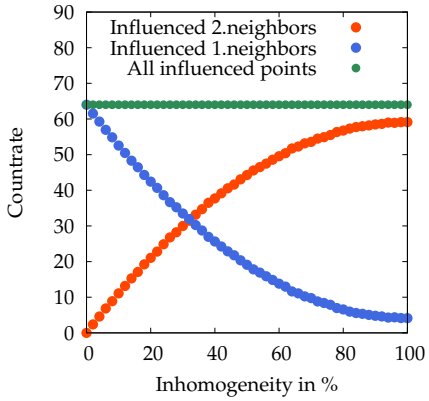


Homogeneous field Inhomogeneous field

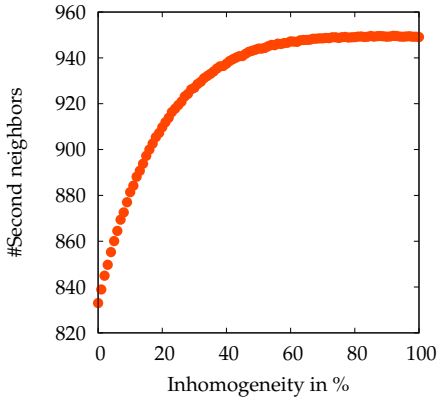
Figure 5.18: Neighborhood relations in homogeneous and inhomogeneous fields. The blue circle represents the chosen center point, the red dots are its direct neighbors and the green dots the second neighbors. The arrows illustrate how the points influence each other. bidirectional arrow means mutual influence, while directed arrows point in the direction of the point that is influenced by the point from which the arrow starts.

influence is not possible here. For inhomogeneous fields the amount of such points increases and the 2-step interaction gets important. As the average number of influenced points for each point has to $2(q+1) = 64$, not much higher neighbors than first and second contribute to the evolution process of a given point.[13] As easy as this may sound the connections between these first and second neighbors make the problem extremely complicated, biquadric fields are strongly connected, what can also be seen if we calculate the average number of second neighbors each point has. Fig. 5.19(b) shows that the number of second neighbors increases with increasing inhomogeneity. For completely random fields almost all points of the affine plane are second neighbors of a given point. Therefore the interaction between nearly all points of the affine plane influences each point separately in the Game of Life, showing the immense complexity of this model on

[13]It could maybe be that for some biquadric fields also third neighbors are important.



(a) Properties of first and second neighbors in biquadric fields under increasing inhomogeneity for prime number $q = 31$. Blue are the symmetric points and red the influenced second neighbors.



(b) Number of second neighbors each point has in average for prime number $q = 31$.

Figure 5.19: Properties of neighborhood relations of biquadric fields under increasing inhomogeneity for $q = 31$

biquadric backgrounds. But the results confirm J. Höfers results on the diameter of biquadric fields, deduced from graph theory ([Höf18]).

6 Conclusion and Outlook

> "The most remarkable property of the universe is that is has spawned creatures able to ask questions"
>
> Stephen Hawking, *The Theory of Everything*

The topic of biquadric fields is a completely new topic to both physics and mathematics, therefore the main scope of this master's thesis was to further investigate biquadrics and the corresponding fields, as well as to get some intuition on how they connect points. This knowledge is important for a physical world based on finite projective geometries because biquadrics encode the most basic physical quantities as length, distance and neighborhood. Furthermore the key of connecting the finite model with existing continuous theories like GR is to establish a connection between the behavior of the metric tensor field on the one hand and the biquadric field on the other hand. One idea to do this, is to impose symmetry conditions on biquadric fields and show that in the continuum limit the Einstein equations are reproduced - how exactly this limit can be taken (this would demand for a topology in the finite space) would be an important topic for further thesis and it shows the tremendous importance of understanding biquadrics correctly. ([Mec18], p. 40-41)
To make connection to GR it would also be important to define a notion of curvature. A first idea was given in chapter 4 in terms of flat(homogeneous) and curved(inhomogeneous) biquadric fields. It was shown that finding unique[1] flat biquadric fields in the whole projective plane is still an open topic - this thesis showed that such fields exist for $\mathbb{P}^2\mathbb{F}_3$, but did not find

[1] In the sense that the center point of the biquadric is unique.

any flat fields for higher prime numbers. If only the affine space is taken into account it is easy to generate flat and symmetric fields by translation of biquadrics. Another important topic is to see whether the point set of a biquadric determines its biquadric representation matrix uniquely, or if there is ambiguity. Indeed it was shown, that for $q > 7$ all representation matrices are uniquely determined by their point set, because no *pseudo Lorentz-Transformation* exists that mixes points of the two quadric partners. A detailed analysis of Lorentz-Transformations on the point set showed, that the transformations can be grouped in classes of point-symmetric, axial-symmetric and rotational ones - this is another topic that should be examined further in the future.
This thesis showed that dynamical systems like the cellular automata *Game of Life* crucially depend on the neighborhood relations, and that their behavior in biquadric neighborhoods differs from standard neighborhood relations. To explain this theoretically one has to take correlations between the points into account, what is done in terms of a cumulant expansion up to second order. The introduction of inhomogeneities into the homogeneous biquadric field changes the behavior again - this can be explained by some kind of phase transition that takes place.
Including the open questions already asked above, there is still much work to be done (for more work see [Mec18], p. 59-63):

- How can the rotating sense of Lorentz-transformations be reasonably defined in finite geometries?

- How can the fundamental exchange transformation of two quadrics be calculated?

- How can a reasonable continuum limit be done? Is possible to reproduce the dynamics of GR from biquadric dynamics in this limit?

- How can we generate flat states for higher prime numbers? Is this even possible? Due to high amount of computation time one has to find different approaches towards this topic.

- How can curvature be denoted precisely in terms of inhomogeneous biquadric fields?

- The local domain seems like a promising way to get rid of order problems in finite geometries - but what exactly is the meaning of the local domain? Is it the real physical space? And which meaning do all the points outside the local domain have in a physical world? There is work on this by Klaus Mecke, which already looks quite promising. ([Mec18], p.39)

Even if this thesis only covered a small part of the whole finite world model, it still achieved progress in the understanding of biquadric fields and their properties, which are indispensable for finding connection to GR and QFT in the continuum limit.

Bibliography

[Ahm65] Y. Ahmavaara. "Structure of Space and the Formalism of Relativistic Quantum Theory". In: *Jour. Math. Phys.* **6** (1965), 87, 220 and 7 (1966) 197, 201.

[Ale12] N. Alex. "Quadriken in endlichen projektiven Ebenen". Bachelor's thesis. FAU Erlangen-Nürnberg, 2012.

[Bat97] L.M. Batten. *Combinatorics of finite geometries*. Second edition. New York: Cambridge university Press, 1997. ISBN: 978-0-521-59993-1.

[Bom+87] L. Bombelli et al. "Space-Time as a causal set". In: *Phys. Rev. Lett.* **59** (1987), pp. 521–524.

[BR04] A. Beutelspacher and U. Rosenbaum. *Projektive Geometrie. Von den Grundlagen bis zu den Anwendungen*. Vieweg+Teubner Verlag, 2004. ISBN: 3-528-17241-X.

[CD98] B. Chopard and M. Droz. *Cellular Automata. Modeling of Physical Systems*. Cambridge University Press, 1998. ISBN: 0-521-46168-5.

[Des70] R. Descartes. *Rene Descartes' philosophische Werke. Principia philosophiae 2.Teil: Über die Prinzipien der körperlichen Dinge*. Trans. by J.H. v. Kirchmann. Berlin: Heimann Verlag, 1870, p. 85.

[Gar70] M. Gardner. "The fantastic combinations of John Conway's new solitaire game 'life'". In: *Scientific American* **223**.4 (Oct. 1970), pp. 120–123.

[Har74] A. B. Harris. "Effect of random defects on the critical behaviour of Ising models". In: *Journal of Physics C: Solid State Physics* **7**.9 (1974), pp. 1671–1692. DOI: 10.1088/0022-3719/7/9/009. URL: http://stacks.iop.org/0022-3719/7/i=9/a=009.

[Höf18] J. Höfer. "Local Domains and Graph Diameter of Biquadric Fields in Finite in a Finite Geometry". Master's thesis, personal communication. FAU Erlangen-Nürnberg, to be published in 2018.

[Jär76] G. Järnefelt. "On the possibility of a finite model describing the universe". In: *Astron. Nachr.* **297**.3 (1976), pp. 131–139. DOI: 10.1002/asna.19762970305.

[JK49] G. Järnefelt and P. Kustaanheimo. *An Observation on Finite Geometries, Den 11te Skandinaviske Matematikerkongress*. Trondheim, 1949, pp. 166–182.

[KB18] P. Knabner and W. Barth. *Lineare Algebra*. Vol. 2: *Grundlagen und Anwendungen*. Springer Spektrum, 2018. ISBN: 978-3-662-55599-6.

[Las14] A. Laska. "Biquadric Fields:Equipping projective spaces with "metric" structure". Master's thesis. FAU Erlangen-Nürnberg, 2014.

[Las19] A. Laska. "Working title: Finite geometries and fundamental physics". PhD thesis, personal communication. FAU Erlangen-Nürnberg, to be published in 2019.

[Mec17] K. Mecke. "Biquadrics configure finite projective geometry into a quantum spacetime". In: *EPL (Europhysics Letters)* **120**.1 (2017). DOI: 10.1209/0295-5075/120/10007.

[Mec18] K. Mecke. *Raum - Zeit - Materie. Auf der Suche nach einer einheitlichen Theorie der fundamentalen Naturgesetze*. 2018. URL: https://theorie1.physik.uni-erlangen.de/media/pdf/publications/finites_weltbild-kurzfassung.pdf.

[Neu66] J. von Neumann. *Theory of Self-Reproducing Automata*. Ed. by A. W. Burks. University of Illinois Press, 1966. ISBN: 9-780-25-272-7337.

[Pen06] R. Penrose. *The road to reality*. New York: Knopf Verlag, 2006. ISBN: 0679454438.

[Rei16] T. Reinhart. "Lorentz-Transformationen in endlichen Raumzeiten". Bachelor's thesis. FAU Erlangen-Nürnberg, 2016.

[Ric11] J. Richter-Gebert. *Perspectives on projective geometry*. Springer Verlag, 2011. ISBN: 978-3-642-17285-4.

[RO09] J. Richter-Gebert and T. Orendt. *Geometriekalküle*. Springer-Verlag, 2009. ISBN: 978-3-642-02529-7.

[Ros09] H.E. Rose. *A course on finite groups*. Springer-Verlag, 2009. ISBN: 978-1-84882-888-9.

[SS78] L.S. Schulman and P.E. Seiden. "Statistical Mechanics of a Dynamical System Based on Conway's Game of Life". In: *Jour. Stat. Phys.* **19**.3 (1978), pp. 293–314. ISSN: 1572-9613. DOI: 10.1007/BF01011727. URL: https://doi.org/10.1007/BF01011727.

[Str12] W. Strampp. *Höhere Mathematik 1. Lineare Algebra*. Vieweg+Teubner Verlag, 2012. ISBN: 978-3-8348-1744-0.

[Ula52] S. Ulam. "Random Processes and Transformations". In: *Proc. Int. Congr. Math.* **2** (1952), pp. 264–275.

[Wey88] H. Weyl. *Raum, Zeit, Materie*. Ed. by J. Ehlers. Berlin: Springer Verlag, 1988.

[Wol83] S. Wolfram. "Statistical mechanics of cellular automata". In: *Rev. Mod. Phys.* **55** (1983), p. 601.

Danksagung

Am Ende dieser Masterarbeit und damit auch am Ende meines Physikstudiums ist es an der Zeit Danke zu sagen. Zu allererst und ganz besonders bei meiner Familie, die mich im Laufe des Studiums und während der Masterarbeit immer bestärkt hat und mir ein wichtiger Rückhalt ist und war.
Zudem möchte ich Klaus Mecke danken, auf dessen Ideen das gesamte Projekt einer finiten projektiven Quantengravitation aufbaut, der mir bei Fragen immer zur Verfügung stand und mich auf meinem Weg durch die Masterarbeit anleitete. Nicht weniger dankbar bin ich aber auch Alexander Laska, der mir immer wieder helfend zur Seite stand, wenn Probleme auftraten, oder ich gerade nicht wusste, wie ich weiter machen sollte. Bei ihm bedanke ich mich auch für das Korrektur lesen der Masterarbeit, und die viele Zeit, die er investierte.

Vielen Dank!

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Erlangen, den 20. Dezember 2018

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